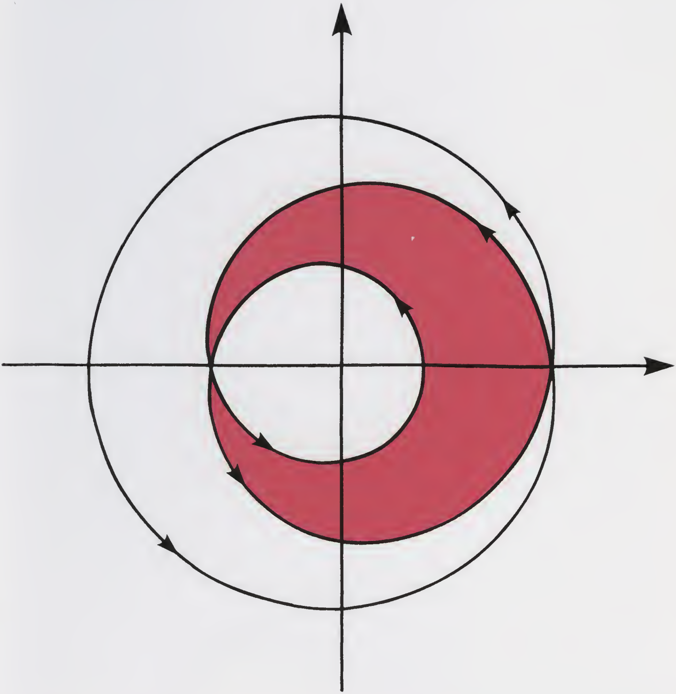
M337 Unit C2
Mathematics: A Third Level Course

The Open University

COMPLEX ANALYSIS

UNIT C2 ZEROS AND EXTREMA



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M337 Unit C2
Mathematics: A Third Level Course

COMPLEX ANALYSIS

UNIT C2 ZEROS AND EXTREMA

Prepared by the Course Team

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Before working through this text, make sure that you have read the *Course Guide* for M337 Complex Analysis.

The Open University, Walton Hall, Milton Keynes, MK7 6AA.

First published 1993. Reprinted 1995, 1999, 2003, 2006

Edited, designed and typeset by the Open University using the Open University TEX System.

Printed in Malta by Gutenberg Press Limited.

ISBN 0 7492 2184 4

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1.4

CONTENTS

INTRODUCTION

In this unit we investigate two practical questions about analytic functions.

> What can we say about the number and location of the zeros of a given analytic function?

For example, how many zeros does the function

$$f(z) = z^5 + z^3 + iz + 1$$

See Problem 2.4(a).

have in the open disc $\{z : |z| < 2\}$?

> How do we determine the maximum (or minimum) value taken by the modulus of a given analytic function on a compact set in the domain of the function?

For example, what is the maximum value of

$$|f(z)| = |z^2 - 4z - 3| \text{ on the set } \{z : |z| \leq 2\}?$$

See Example 4.1.

In the process of developing methods for answering these questions, we shall encounter more theoretical questions such as:

> What does the image of a region under an analytic function look like?

For example, is it also a region?

In Section 1 we introduce a geometric notion called the *winding number* of a closed path Γ round a point α. This counts the number of times that the path Γ winds round the point α. We then show that if Γ is a closed contour, then the winding number of Γ round α can be expressed as a certain contour integral around Γ.

In Section 2 we use the Residue Theorem to prove a key result, called the Argument Principle, which relates the number of zeros of an analytic function f inside a simple-closed contour Γ to the winding number of $f(\Gamma)$ round 0. This leads to various methods for finding the number of zeros of f inside Γ. One such method, based on a result called Rouché's Theorem, leads to a proof of the Fundamental Theorem of Algebra.

Section 3 is devoted to a number of theoretical consequences of the Argument Principle, which concern the local behaviour of analytic functions. For example, the Open Mapping Theorem states that non-constant analytic functions always map open sets to open sets.

Finally, in Section 4, we use the Open Mapping Theorem to obtain a result called the Maximum Principle. This result simplifies the search for the maximum value of the modulus of an analytic function on a compact set, because it tells us that this maximum value must be attained somewhere on the boundary of the set.

Study guide

Section 1 contains a number of new concepts, such as the winding number and a family of logarithm functions, which are needed both in this unit and the next. You should aim to become familiar with these ideas before proceeding further. In particular, the winding number plays a key role in Section 2, which contains the audio-tape subsection on Rouché's Theorem.

Sections 3 and 4 are rather more theoretical, but they both contain results which will be needed later in the course.

1 THE WINDING NUMBER

After working through this section, you should be able to:

(a) define the *winding number of a path* round a point and determine it in simple cases;

(b) define a *generalized argument function* and *logarithm function* and their associated cut planes, and evaluate such functions;

(c) express the *winding number of a closed contour* round a point as a contour integral.

1.1 What is the winding number?

Imagine that a dog is tied by its lead to a post, but is otherwise free to roam. The path of the dog during a given time is shown in Figure 1.1. The dog's initial point is the same as its final point, so that the path is closed. It seems that the dog has wandered around paying particular attention to a certain stone and also trying (unsuccessfully) to reach a tree.

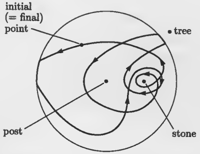


Figure 1.1

In the course of its journey, the dog went twice round the post, in an anticlockwise direction. You can either 'see' this or else, more systematically, count the number of times that the path crosses (in an anticlockwise direction) the straight line indicating the location of the post. A similar approach can be used to count how many times the dog went round, or *wound* round, any other point which lies off the path. Simply consider a ray from the given point and count how many times the path crosses the ray; anticlockwise crossings count as $+1$, whereas clockwise crossings count as -1.

Problem 1.1

How many times did the dog wind round the following points?

(a) the stone (b) the tree

We frequently encounter paths Γ which are specified by means of their parametrizations γ, and it is then not so easy to determine from γ how many times the path winds round a given point. We want to develop a method of doing this and we begin with a simple example of a path Γ which winds round the point 0.

Consider the closed path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \qquad (t \in [0, 2\pi]),$$

illustrated in Figure 1.2.

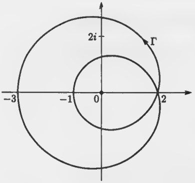


Figure 1.2

t	$\gamma(t)$
0	2
$\pi/4$	$(2+1/\sqrt{2})i$
$\pi/2$	-3
$3\pi/4$	$-(2+1/\sqrt{2})i$
π	2
$5\pi/4$	$(2-1/\sqrt{2})i$
$3\pi/2$	-1
$7\pi/4$	$-(2-1/\sqrt{2})i$
2π	2

Imagine what happens to the image point $\gamma(t)$ as the parameter t increases from 0 to 2π. It traverses Γ, starting and finishing at the point $\gamma(0) = \gamma(2\pi) = 2$, winding twice round 0 (anticlockwise) in the process.

The manner in which a path such as Γ winds round 0 depends on the 'angular change' of the ray from 0 through $\gamma(t)$ which occurs as $\gamma(t)$ traverses the path Γ. To measure this 'angular change' for a given path $\Gamma : \gamma(t)$ $(t \in [a, b])$, we need to choose an argument $\theta(t)$ for each point $\gamma(t)$ on Γ in such a way that the argument $\theta(t)$ varies continuously as t increases from a to b. The 'angular change' along Γ is then equal to the change in the 'argument function'

$$\theta : t \longmapsto \theta(t) \qquad (t \in [a, b]),$$

given by $\theta(b) - \theta(a)$. Notice that we cannot simply choose $\theta(t) = \operatorname{Arg}(\gamma(t))$ in general, since this function would jump by 2π as $\gamma(t)$ crossed the negative real axis, and so fail to be continuous (see Figure 1.3). However, if Γ does not meet this axis, then the choice $\theta(t) = \operatorname{Arg}(\gamma(t))$ is possible.

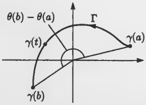


Figure 1.3

> **Definition** Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path in $\mathbb{C} - \{0\}$. A **continuous argument function** for Γ is a continuous function
>
> $$\theta : [a, b] \longrightarrow \mathbb{R}$$
>
> such that, for each $t \in [a, b]$, $\theta(t)$ is an argument of $\gamma(t)$.

Note that Γ need not be a closed path in this definition.

Since $\theta(t)$ is an argument of $\gamma(t)$, we can write

$$\gamma(t) = |\gamma(t)|e^{i\theta(t)}, \qquad \text{for } t \in [a, b],$$

and so

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta(t)}, \qquad \text{for } t \in [a, b]. \tag{1.1}$$

For a given path $\Gamma : \gamma(t)$ $(t \in [a, b])$, therefore, we obtain a continuous argument function by expressing $\gamma(t)/|\gamma(t)|$ in the form $e^{i\theta(t)}$, where θ is a continuous function on $[a, b]$.

Example 1.1

Determine a continuous argument function for the path

$$\Gamma : \gamma(t) = 2e^{it} \qquad (t \in [0, \pi]).$$

Solution

Since $|\gamma(t)| = 2$, for $t \in [0, \pi]$, we have

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{it}, \qquad \text{for } t \in [0, \pi].$$

Thus one choice of continuous argument function is

$$\theta(t) = t \qquad (t \in [0, \pi]). \quad \blacksquare$$

Remark The solution to Example 1.1 is not unique. Indeed, we could have chosen $\theta(t) = t + 2\pi$ or, in general

$$\theta(t) = t + 2n\pi \qquad (t \in [0, \pi]),$$

for any fixed $n \in \mathbb{Z}$, since θ is continuous on $[0, \pi]$ and

$$e^{i\theta(t)} = e^{i(t+2n\pi)} = e^{it}, \qquad \text{for } t \in [0, \pi].$$

Note that these choices of continuous argument function all differ by integer multiples of 2π.

Example 1.2

Determine a continuous argument function for the path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \qquad (t \in [0, 2\pi]),$$

illustrated in Figure 1.2.

Solution

Since $2 + \sin t > 0$, for $t \in [0, 2\pi]$, we have

$$|\gamma(t)| = |2 + \sin t| = 2 + \sin t, \qquad \text{for } t \in [0, 2\pi],$$

so that

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{2it}, \qquad \text{for } t \in [0, 2\pi].$$

Thus one choice of continuous argument function is

$$\theta(t) = 2t \qquad (t \in [0, 2\pi]). \quad \blacksquare$$

Notice, in Example 1.2, that as t increases from 0 to 2π the image point $\gamma(t)$ starts from $\gamma(0) = 2$ and returns to $\gamma(2\pi) = 2$, the same point; however, the value of the continuous argument function θ increases from $\theta(0) = 0$ to $\theta(2\pi) = 4\pi$, reflecting the fact that the path Γ winds twice around 0.

Problem 1.2

Determine a continuous argument function for each of the following paths.

(a) $\Gamma : \gamma(t) = te^{\pi it} \quad (t \in [1, 3])$

(b) $\Gamma : \gamma(t) = 1 + it \quad (t \in [0, 1])$

(c) $\Gamma : \gamma(t) = e^{-4it} \quad (t \in [0, 2\pi])$

Intuitively it seems clear that every path Γ in $\mathbb{C} - \{0\}$ has a continuous argument function. However, the proof is rather tricky because, as you have seen earlier in the course, paths can be complicated.

Theorem 1.1 Any path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ, which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

The proof of Theorem 1.1 appears at the end of this subsection.

The 'angular change' of $\gamma(t)$ along the path $\Gamma : \gamma(t)$ $(t \in [a, b])$ is given by $\theta(b) - \theta(a)$, and so it follows that Γ winds round 0

$$\frac{1}{2\pi}(\theta(b) - \theta(a)) \text{ times}$$

(because an angular increase of 2π corresponds to one anticlockwise turn round 0). We use this quantity to define the *winding number* of any path (closed or not) round 0.

Definition Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path lying in $\mathbb{C} - \{0\}$. Then the **winding number** of Γ round 0 is

$$\text{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is any continuous argument function for Γ.

Remarks

1 The winding number is well defined because (by Theorem 1.1) any two choices of continuous argument function for Γ differ by an integer multiple of 2π.

2 For an arbitrary path Γ the winding number is a real number but, for a closed path, it is always an integer because $\theta(b)$ and $\theta(a)$ are both arguments of $\gamma(a) = \gamma(b)$ and so differ by an integer multiple of 2π. For example, the path

$$\Gamma : \gamma(t) = 2e^{it} \qquad (t \in [0, \pi]),$$

in Example 1.1, has continuous argument function $\theta(t) = t$ $(t \in [0, \pi])$, so that

$$\text{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\pi - 0) = \frac{1}{2}.$$

On the other hand, the path

$$\Gamma : \gamma(t) = (2 + \sin t)e^{2it} \qquad (t \in [0, 2\pi]),$$

in Example 1.2, has continuous argument function $\theta(t) = 2t$ $(t \in [0, 2\pi])$, so that

$$\text{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(4\pi - 0) = 2.$$

Problem 1.3

Calculate the winding number round 0 of each of the paths Γ in Problem 1.2, by using the continuous argument functions found in Problem 1.2. Check your answers by sketching Γ.

Problem 1.4

Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a path in $\mathbb{C} - \{0\}$, let $c \in]a, b[$ and consider the subpaths $\Gamma_1 : \gamma(t)$ $(t \in [a, c])$ and $\Gamma_2 : \gamma(t)$ $(t \in [c, b])$. Prove that

$$\text{Wnd}(\Gamma, 0) = \text{Wnd}(\Gamma_1, 0) + \text{Wnd}(\Gamma_2, 0).$$

In order to prove Theorem 1.1, we introduce a family of argument functions, which generalize the principal argument function Arg.

Definition For $\phi \in \mathbb{R}$, the function Arg_ϕ is defined by

$$\text{Arg}_\phi(z) = \theta \qquad (z \in \mathbb{C} - \{0\}),$$

where θ is the argument of z lying in the interval $]\phi - 2\pi, \phi]$.

We read $\text{Arg}_\phi(z)$ as 'arg ϕ of z'.

For example,

$$\text{Arg}_\pi(-i) = -\pi/2,$$

since $-\pi/2$ is the argument of $-i$ which lies in $]-\pi, \pi]$, whereas

$$\text{Arg}_{2\pi}(-i) = 3\pi/2,$$

since $3\pi/2$ is the argument of $-i$ which lies in $]0, 2\pi]$ (see Figure 1.4).

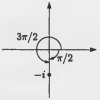


Figure 1.4

Problem 1.5

Evaluate each of the following expressions.

(a) $\text{Arg}_\pi(i)$ (b) $\text{Arg}_0(-1)$ (c) $\text{Arg}_{3\pi/2}(1 - i)$

Since $-\pi < \text{Arg}\, z \leq \pi$, it is evident that the function Arg_π is just the principal argument function Arg. Now Arg is continuous when restricted to the cut plane $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$, and each of these generalized argument functions is continuous on an appropriate cut plane.

Unit A3, Frame 17

Definition For $\phi \in \mathbb{R}$, the cut plane $\mathbb{C}_\phi$ is defined by

$$\mathbb{C}_\phi = \left\{re^{i\theta} : r > 0, \phi - 2\pi < \theta < \phi\right\}.$$

For example,

$$\begin{aligned}\mathbb{C}_\pi &= \left\{re^{i\theta} : r > 0, -\pi < \theta < \pi\right\} \\ &= \mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}\end{aligned}$$

(see Figure 1.5), and

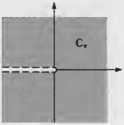


Figure 1.5

$$\begin{aligned}\mathbb{C}_{2\pi} &= \left\{re^{i\theta} : r > 0, 0 < \theta < 2\pi\right\} \\ &= \mathbb{C} - \{x \in \mathbb{R} : x \geq 0\}\end{aligned}$$

(see Figure 1.6).

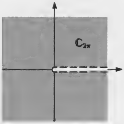


Figure 1.6

Notice that

$$\mathbb{C}_{\phi + 2n\pi} = \mathbb{C}_\phi, \qquad \text{for } n \in \mathbb{Z},$$

since any two arguments of a complex number differ by an integer multiple of 2π. For example,

$$\ldots = \mathbb{C}_{-\pi} = \mathbb{C}_\pi = \mathbb{C}_{3\pi} = \mathbb{C}_{5\pi} = \ldots.$$

Problem 1.6

Sketch each of the following cut planes.

(a) $\mathbb{C}_0$ (b) $\mathbb{C}_{3\pi/2}$

> **Theorem 1.2** For all $\phi \in \mathbb{R}$, Arg_ϕ is continuous on $\mathbb{C}_\phi$.

Proof By definition, $\mathrm{Arg}_\phi(z)$ is that argument of z which lies in $]\phi - 2\pi, \phi]$. Therefore $\mathrm{Arg}_\phi(z) - \phi + \pi$ lies in $]-\pi, \pi]$ and is an argument of

$$ze^{-i\phi}e^{i\pi} = -ze^{-i\phi}.$$

Hence

$$\mathrm{Arg}_\phi(z) - \phi + \pi = \mathrm{Arg}(-ze^{-i\phi}),$$

which gives

$$\mathrm{Arg}_\phi(z) = \phi - \pi + \mathrm{Arg}(-ze^{-i\phi}), \qquad \text{for } z \in \mathbb{C} - \{0\}. \tag{1.2}$$

Now, if $z \in \mathbb{C}_\phi$, then $-ze^{-i\phi} = ze^{-i\phi}e^{i\pi} \in \mathbb{C}_\pi$ and so the continuity of the function Arg_ϕ on $\mathbb{C}_\phi$ follows, using Equation (1.2), from that of Arg on $\mathbb{C}_\pi$. ■

By Theorem 1.2, it is easy to determine a continuous argument function for a path $\Gamma : \gamma(t)$ $(t \in [a, b])$ which lies entirely in a cut plane $\mathbb{C}_\phi$ (see Figure 1.7), since we can simply take

$$\theta(t) = \mathrm{Arg}_\phi(\gamma(t)) \qquad (t \in [a, b]).$$

Therefore, one way to obtain a continuous argument function for a path Γ is to break Γ up into subpaths, each of which lies in a cut plane, and then choose continuous argument functions for the subpaths, which match up at points where the subpaths join. This is the idea behind the proof of Theorem 1.1.

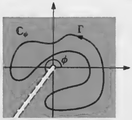


Figure 1.7

> **Theorem 1.1** Any path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ, which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

Proof Given any path $\Gamma : \gamma(t)$ $(t \in [a, b])$ lying in $\mathbb{C} - \{0\}$, we can apply the Paving Theorem to partition the interval $[a, b]$ into adjacent subintervals $[t_{k-1}, t_k]$, $k = 1, 2, \ldots, n$, so that

Unit B1, Theorem 3.7

$$\gamma([t_{k-1}, t_k]) \subseteq D_k, \quad \text{for } k = 1, 2, \ldots, n,$$

where each D_k is an open disc in $\mathbb{C} - \{0\}$ (see Figure 1.8).

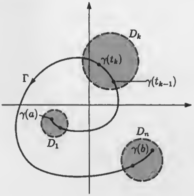


Figure 1.8

If D_k has centre $r_k e^{i\theta_k}$, then $D_k \subseteq \mathbb{C}_{\phi_k}$, where $\phi_k = \theta_k + \pi$, and so the argument function Arg_{ϕ_k} is continuous on D_k. Thus, for $k = 1, 2, \ldots, n$, $\text{Arg}_{\phi_k}(\gamma(t))$ is a continuous argument function for the subpath

$$\Gamma_k : \gamma(t) \qquad (t \in [t_{k-1}, t_k])$$

(see Figure 1.9).

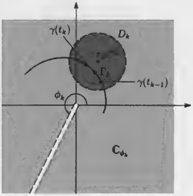


Figure 1.9

To ensure that these argument functions match up at the points $\gamma(t_k)$, for $k = 1, 2, \ldots, n-1$, we need to choose the arguments θ_k so that

$$\text{Arg}_{\phi_{k+1}}(\gamma(t_k)) = \text{Arg}_{\phi_k}(\gamma(t_k)), \qquad \text{for } k = 1, 2, \ldots, n-1.$$

This is possible since $\gamma(t_k) \in D_k$ and $\gamma(t_k) \in D_{k+1}$.

It then follows that

$$\theta(t) = \begin{cases} \text{Arg}_{\phi_1}(\gamma(t)), & t_0 \le t \le t_1, \\ \text{Arg}_{\phi_2}(\gamma(t)), & t_1 \le t \le t_2, \\ \vdots & \\ \text{Arg}_{\phi_n}(\gamma(t)), & t_{n-1} \le t \le t_n, \end{cases}$$

is a continuous argument function for Γ. ■

1.2 The winding number as a contour integral

The winding number, which is a very natural geometric concept, is related to complex integration. In order to see the connection, we introduce a family of logarithm functions.

Definition For $\phi \in \mathbb{R}$, the function Log_ϕ is defined by

$$\text{Log}_\phi(z) = \log_e |z| + i \,\text{Arg}_\phi(z) \qquad (z \in \mathbb{C} - \{0\}).$$

We read $\text{Log}_\phi(z)$ as 'log ϕ of z', in contrast to 'log to the base e of x' for $\log_e x$.

For example, since $\text{Arg}_\pi = \text{Arg}$, the function Log_π is just the principal logarithm function:

$$\text{Log}_\pi = \text{Log}.$$

Note that if $z \in \mathbb{C} - \{0\}$, then $\text{Log}_\phi(z)$ *is* a logarithm of z, since

$$\begin{aligned} \exp(\text{Log}_\phi(z)) &= \exp(\log_e |z| + i \,\text{Arg}_\phi(z)) \\ &= |z| e^{i \,\text{Arg}_\phi(z)} \\ &= z, \end{aligned}$$

because $\text{Arg}_\phi(z)$ is an argument of z.

Moreover, we can show that the function Log_ϕ is analytic on the cut plane $\mathbb{C}_\phi$ and that its derivative has the same rule as that of the principal logarithm function Log.

Theorem 1.3 For all $\phi \in \mathbb{R}$, the function Log_ϕ is analytic on $\mathbb{C}_\phi$ with derivative Log'_ϕ given by

$$\mathrm{Log}'_\phi(z) = \frac{1}{z}, \qquad \text{for } z \in \mathbb{C}_\phi.$$

Proof For $z \in \mathbb{C}_\phi$ we have, by Equation (1.2),

$$\begin{aligned}
\mathrm{Log}_\phi(z) &= \log_e |z| + i\,\mathrm{Arg}_\phi(z) \\
&= \log_e |z| + i(\phi - \pi + \mathrm{Arg}(-ze^{-i\phi})) \\
&= \log_e |-ze^{-i\phi}| + i(\phi - \pi + \mathrm{Arg}(-ze^{-i\phi})) \\
&= i(\phi - \pi) + \mathrm{Log}(-ze^{-i\phi}),
\end{aligned}$$

and also $-ze^{-i\phi} \in \mathbb{C}_\pi$. Thus, by the Chain Rule,

$$\begin{aligned}
\mathrm{Log}'_\phi(z) &= 0 + \frac{-e^{-i\phi}}{-ze^{-i\phi}} \\
&= \frac{1}{z}, \qquad \text{for } z \in \mathbb{C}_\phi,
\end{aligned}$$

as required. ■

Using these generalized logarithm functions we can represent the winding number of a closed contour round 0 as a contour integral.

Theorem 1.4 Let $\Gamma : \gamma(t)$ $(t \in [a, b])$ be a closed contour lying in $\mathbb{C} - \{0\}$. Then

$$\mathrm{Wnd}(\Gamma, 0) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{z}\, dz.$$

Note that Γ has to be a *contour*, as opposed to a path, for the integral to be defined.

Proof First we apply the Paving Theorem as in the proof of Theorem 1.1, retaining the same notation. Since the disc $D_k \subseteq \mathbb{C}_{\phi_k}$, we deduce that the function $f(z) = 1/z$ has primitive $F(z) = \mathrm{Log}_{\phi_k}(z)$ on D_k, by Theorem 1.3. Hence, by the Fundamental Theorem of Calculus,

Unit B1, Theorem 3.1

$$\begin{aligned}
\int_{\Gamma_k} \frac{1}{z}\, dz &= \mathrm{Log}_{\phi_k}(\gamma(t_k)) - \mathrm{Log}_{\phi_k}(\gamma(t_{k-1})) \\
&= (\log_e |\gamma(t_k)| + i\,\mathrm{Arg}_{\phi_k}(\gamma(t_k))) \\
&\quad - (\log_e |\gamma(t_{k-1})| + i\,\mathrm{Arg}_{\phi_k}(\gamma(t_{k-1}))) \\
&= \log_e |\gamma(t_k)| - \log_e |\gamma(t_{k-1})| + i(\theta(t_k) - \theta(t_{k-1})),
\end{aligned}$$

Here θ is the continuous argument function for Γ produced in the proof of Theorem 1.1.

for $k = 1, 2, \ldots, n$. Thus

$$\begin{aligned}
\int_\Gamma \frac{1}{z}\, dz &= \sum_{k=1}^{n} \int_{\Gamma_k} \frac{1}{z}\, dz \\
&= \sum_{k=1}^{n} (\log_e |\gamma(t_k)| - \log_e |\gamma(t_{k-1})|) + i \sum_{k=1}^{n} (\theta(t_k) - \theta(t_{k-1})) \\
&= (\log_e |\gamma(b)| - \log_e |\gamma(a)|) + i(\theta(b) - \theta(a)) \quad \text{(telescoping cancellation)} \\
&= i(\theta(b) - \theta(a)) \quad (\text{since } \gamma(a) = \gamma(b)) \\
&= 2\pi i\, \mathrm{Wnd}(\Gamma, 0),
\end{aligned}$$

as required. ■

For example, if Γ is the unit circle, then

$$\mathrm{Wnd}(\Gamma,0) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z}\,dz = 1, \text{ as expected.}$$

Although Theorem 1.4 can be useful for evaluating contour integrals of the form $\int_\Gamma 1/z\,dz$, its main use is to give certain other contour integrals a geometric interpretation (see, in particular, Section 2).

1.3 The winding number round an arbitrary point

If $\Gamma : \gamma(t)$ $(t \in [a,b])$ is a path and α is a point not lying on Γ, then we can define the winding number $\mathrm{Wnd}(\Gamma,\alpha)$ of Γ round α, in a similar way to $\mathrm{Wnd}(\Gamma,0)$. We introduce a **continuous argument function θ_α for Γ relative to α** (that is, θ_α is continuous on $[a,b]$ and $\theta_\alpha(t)$ is an argument of $\gamma(t)-\alpha$) and then define the **winding number of Γ round α** by

$$\mathrm{Wnd}(\Gamma,\alpha) = \frac{1}{2\pi}(\theta_\alpha(b) - \theta_\alpha(a)).$$

Alternatively, we can introduce the idea of a **translated path**:

$$\Gamma - \alpha : \gamma(t) - \alpha \qquad (t \in [a,b]), \tag{1.3}$$

(see Figure 1.10), and then define

$$\mathrm{Wnd}(\Gamma,\alpha) = \mathrm{Wnd}(\Gamma - \alpha, 0). \tag{1.4}$$

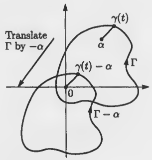


Figure 1.10

As before, the winding number round α can be determined by inspection if Γ is a given closed path which does not pass through α.

Problem 1.7

Determine by inspection the winding number of the closed path Γ in Figure 1.2 of Example 1.2 round each of the points $1, -2, -2i, 3i$.

Problem 1.8

Prove that if Γ is a closed contour and α does not lie on Γ then

$$\mathrm{Wnd}(\Gamma,\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-\alpha}\,dz.$$

(*Hint*: Use Theorem 1.4 together with Equations (1.3) and (1.4).)

Finally we give a result about the variation of $\mathrm{Wnd}(\Gamma,\alpha)$ as α varies. It is geometrically evident that if Γ is a closed path then the winding number $\mathrm{Wnd}(\Gamma,\alpha)$ is constant on any open disc D lying in the complement of Γ (see Figure 1.11). We shall need this result later in the unit.

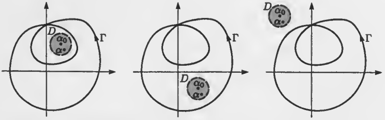


Figure 1.11 $\mathrm{Wnd}(\Gamma,\alpha) = \mathrm{Wnd}(\Gamma,\alpha_0)$

Theorem 1.5 Let Γ be a closed path and let D be an open disc with centre α_0 lying in the complement of Γ. Then the function $\alpha \longmapsto \operatorname{Wnd}(\Gamma, \alpha)$ is constant on D.

Proof First suppose that θ_{α_0} is a continuous argument function for the path $\Gamma : \gamma(t)$ $(t \in [a,b])$ relative to α_0; that is, θ_{α_0} is continuous on $[a,b]$ and, for each $t \in [a,b]$,

$$\theta_{\alpha_0}(t) \text{ is an argument of } \gamma(t) - \alpha_0.$$

We shall use θ_{α_0} to construct a continuous argument function θ_α for Γ relative to α, where $\alpha \in D$. To do this, note that, for $t \in [a,b]$,

$$\frac{\gamma(t)-\alpha}{\gamma(t)-\alpha_0} = 1 - \frac{\alpha-\alpha_0}{\gamma(t)-\alpha_0},$$

so that

$$\gamma(t) - \alpha = (\gamma(t)-\alpha_0)\left(1 - \frac{\alpha-\alpha_0}{\gamma(t)-\alpha_0}\right). \tag{1.5}$$

Now if D has radius r, then

$$|\alpha - \alpha_0| < r \le |\gamma(t) - \alpha_0|, \qquad \text{for } t \in [a,b],$$

so that

$$\left|\frac{\alpha-\alpha_0}{\gamma(t)-\alpha_0}\right| < 1, \qquad \text{for } t \in [a,b];$$

hence $1 - (\alpha-\alpha_0)/(\gamma(t)-\alpha_0)$ lies in $\{z : |z-1| < 1\}$ and so in the cut plane $\mathbb{C}_\pi$ (see Figure 1.12). Thus the function

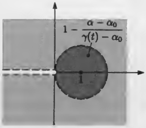


Figure 1.12

$$t \longmapsto \operatorname{Arg}\left(1 - \frac{\alpha-\alpha_0}{\gamma(t)-\alpha_0}\right) \qquad (t \in [a,b])$$

is continuous and so, by Equation (1.5), the function

$$\theta_\alpha(t) = \theta_{\alpha_0}(t) + \operatorname{Arg}\left(1 - \frac{\alpha-\alpha_0}{\gamma(t)-\alpha_0}\right) \qquad (t \in [a,b]) \tag{1.6}$$

A sum of arguments is an argument of the product (*Unit A1*, page 20).

is a continuous argument function for Γ relative to α.

Since $\gamma(a) = \gamma(b)$,

$$\operatorname{Arg}\left(1 - \frac{\alpha-\alpha_0}{\gamma(b)-\alpha_0}\right) = \operatorname{Arg}\left(1 - \frac{\alpha-\alpha_0}{\gamma(a)-\alpha_0}\right),$$

and so, by Equation (1.6),

$$\theta_\alpha(b) - \theta_\alpha(a) = \theta_{\alpha_0}(b) - \theta_{\alpha_0}(a),$$

which shows that

$$\operatorname{Wnd}(\Gamma, \alpha) = \operatorname{Wnd}(\Gamma, \alpha_0),$$

as required. ■

2 LOCATING ZEROS OF ANALYTIC FUNCTIONS

After working through this section, you should be able to:

(a) understand and apply the Argument Principle;

(b) understand Rouché's Theorem and apply it in simple cases to determine the number of zeros of an analytic function in a specified region;

(c) understand how the Fundamental Theorem of Algebra follows from Rouché's Theorem.

2.1 The Argument Principle

We now begin to investigate the location of the zeros of an analytic function and we recall that these zeros are all isolated. The key to the main result of this subsection — the Argument Principle — relates the zeros of an analytic function f to the poles of the related function f'/f.

Unit B3, Theorem 5.3

The function f'/f is called the **logarithmic derivative** of f since
$$\frac{d}{dz}\mathrm{Log}(f(z)) = \frac{f'(z)}{f(z)}.$$

Example 2.1

Let $f(z) = (z-i)^2(z+1)^3$. Determine the poles of the function f'/f and relate them to the zeros of f.

Solution

Since
$$f'(z) = 2(z-i)(z+1)^3 + (z-i)^2\,3(z+1)^2,$$
we have
$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{2(z-i)(z+1)^3 + (z-i)^2\,3(z+1)^2}{(z-i)^2(z+1)^3}\\ &= \frac{2}{z-i} + \frac{3}{z+1}.\end{aligned}$$

Thus f'/f has a simple pole at i with residue 2, whereas f has a zero at i of order two. Also f'/f has a simple pole at -1 with residue 3, whereas f has a zero at -1 of order three. ■

Unit B4, Theorem 2.2

Problem 2.1

Let $f(z) = z^{10}(z-1)$. Determine the poles of the function f'/f and relate them to the zeros of f.

The solutions to Example 2.1 and Problem 2.1 suggest a connection between the order of a zero of f and the residue at the corresponding pole of f'/f. We now establish this connection.

Theorem 2.1 Let an analytic function f have a zero of order n at α. Then the function f'/f has a simple pole at α with

$$\mathrm{Res}(f'/f, \alpha) = n. \qquad (2.1)$$

Proof By assumption

$$f(z) = (z-\alpha)^n g(z),$$

Unit B3, Theorem 5.1

say, where the function g is analytic on some open disc D with centre α, and $g(\alpha) \neq 0$. Therefore

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{n(z-\alpha)^{n-1}g(z) + (z-\alpha)^n g'(z)}{(z-\alpha)^n g(z)}\\ &= \frac{n}{z-\alpha} + \frac{g'(z)}{g(z)}.\end{aligned}$$

Since g is analytic at α and $g(\alpha) \neq 0$, it follows that g'/g is analytic at α. Hence f'/f has a simple pole at α with residue n, as required. ■

Suppose now that a function f is analytic on a simply-connected region $\mathcal{R}$ and that Γ is a simple-closed contour in $\mathcal{R}$ on which f is non-zero. Then, by the Residue Theorem,

Unit C1, Theorem 2.1

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz$$

is the sum of the residues of f'/f at the singularities of f'/f lying inside Γ. Since f' is analytic (because f is), f'/f is analytic at any point of $\mathcal{R}$ where f is non-zero, and so, by Theorem 2.1, the only singularities of f'/f in $\mathcal{R}$ are simple poles at the zeros of f. Therefore, if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the zeros of f lying inside Γ (see Figure 2.1) and if their orders are $n_1, n_2, \ldots, n_k$, respectively, then, by Equation (2.1),

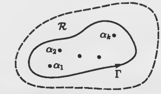


Figure 2.1

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz &= n_1 + n_2 + \cdots + n_k\\ &= N, \end{aligned} \tag{2.2}$$

where N is the number of zeros of f inside Γ, counted according to their orders. For example, if $f(z) = z^3$ and Γ is the unit circle $\{z : |z| = 1\}$, then

$$\begin{aligned}\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_\Gamma \frac{3z^2}{z^3}\,dz\\ &= \frac{3}{2\pi i}\int_\Gamma \frac{1}{z}\,dz\\ &= 3.\end{aligned}$$

It follows from Equation (2.2) that the number of zeros of the function $f(z) = z^3$ lying inside Γ is three, counted according to their orders; this is as expected, since f has a zero of order three at 0 which lies inside Γ.

The importance of Equation (2.2) lies in the geometric interpretation of the integral of f'/f along Γ, which turns out to be the winding number of the image path $f(\Gamma)$ round 0. Thus we obtain the following result.

Theorem 2.2 Argument Principle

Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simple-closed contour in $\mathcal{R}$, such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$$\mathrm{Wnd}(f(\Gamma), 0) = N, \tag{2.3}$$

where N is the number of zeros of f inside Γ, counted according to their orders.

When counting zeros according to their orders, the word *multiplicity* is often used in place of the word *order*.

Proof By the above discussion, it is sufficient to prove that $\text{Wnd}(f(\Gamma),0)$ is equal to $\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{f'(z)}{f(z)}\,dz$. To do this, first let the path Γ be $\Gamma:\gamma(t)\ (t\in[a,b])$; then

Strictly speaking, we should first split Γ into its constituent smooth paths:

$$\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n,$$

as in *Unit B1*.

$$\begin{aligned}
\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{1}{f(\gamma(t))}(f\circ\gamma)'(t)\,dt \quad \text{(by the Chain Rule)} \\
&= \frac{1}{2\pi i}\int_{f(\Gamma)} \frac{1}{w}\,dw \\
&= \text{Wnd}(f(\Gamma),0),
\end{aligned}$$

since $t \longmapsto f(\gamma(t))\ (t\in[a,b])$ is the parametrization of $f(\Gamma)$.

Actually, this argument is valid only if f' is non-zero on Γ, for then the image of each constituent smooth path of Γ is a constituent smooth path of $f(\Gamma)$, so that $f(\Gamma)$ is a contour. In general, however, f' can have zeros on Γ, but only finitely many. Thus, by making small modifications to Γ near the zeros of f' (see Figure 2.2), we can obtain a simple-closed contour Γ' on which f' does *not* vanish and for which

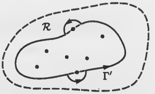


Figure 2.2 Avoiding the zeros of f' on Γ

1. Γ' surrounds the same zeros of f as does Γ;
2. $\text{Wnd}(f(\Gamma'),0) = \text{Wnd}(f(\Gamma),0)$.

Since Equation (2.3) does hold for Γ', it must also hold for Γ. ■

By applying the Argument Principle to the function $f-\beta$ instead of f, we obtain this useful corollary.

> **Corollary** Let a function f be analytic on a simply-connected region $\mathcal{R}$ and let Γ be a simple-closed contour in $\mathcal{R}$, such that $f(z)\neq\beta$, for $z\in\Gamma$. Then $\text{Wnd}(f(\Gamma),\beta)$ is the number of zeros of the function $f-\beta$ inside Γ, counted according to their orders.

Remark Each zero of $f-\beta$ is a solution of the equation $f(z)=\beta$.

As an example, consider the polynomial function $f(z)=z^3-z^2$ and the closed contour $\Gamma=\{z:|z|=2\}$. We have plotted the image $f(\Gamma)$ in Figure 2.3, with the help of a computer.

This image path $f(\Gamma)$ should be familiar — it is the cover design for Block C!

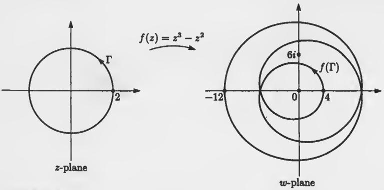


Figure 2.3

Notice first that $f(z)\neq 0$ for $z\in\Gamma$, and that $f(\Gamma)$ winds three times round 0, which implies, by Theorem 2.2, that f has three zeros inside Γ. This is no surprise, since

Notice that we have omitted the phrase 'counted according to their orders' here, but it should always be understood.

$$\begin{aligned}
f(z)=0 \quad &\Longleftrightarrow \quad z^3-z^2=0 \\
&\Longleftrightarrow \quad z^2(z-1)=0,
\end{aligned}$$

so that f has a simple zero at 1 and a zero of order two at 0, both inside Γ.

A less obvious result is obtained by noting that $f(\Gamma)$ winds twice round the point $6i$ (see Figure 2.3), so that, by the corollary, f takes the value $6i$ twice inside Γ. This means that the equation $f(z) = 6i$ (that is, $z^3 - z^2 - 6i = 0$) has two solutions inside Γ, which is not at all obvious.

Problem 2.2

Let $f(z) = z^3 - z^2$ and $\Gamma = \{z : |z| = 2\}$. How many solutions does the equation $f(z) = -1$ have inside Γ?

2.2 Rouché's Theorem (audio-tape)

In the previous subsection we saw that an accurate diagram of the image $f(\Gamma)$ of a simple-closed contour Γ under an analytic function f gives a great deal of information about solutions of equations of the form $f(z) = \beta$. Since such a diagram is not always easy to obtain, it is useful to have other methods to give information about zeros of functions.

We discuss one such method on the audio tape which follows. To prepare for this, you should attempt the following problem, which is closely related to Problem 2.2.

Problem 2.3

Figure 2.4 shows the image of the circle $\Gamma = \{z : |z| = 2\}$ under the function

$$f(z) = z^3 - z^2 + 1.$$

On the basis of this figure, write down the number of zeros of f in $\{z : |z| < 2\}$.

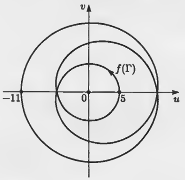
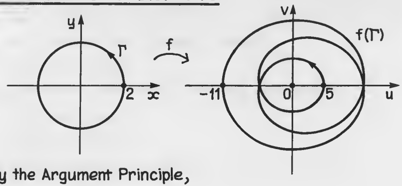


Figure 2.4

NOW START THE TAPE.

1. Solution to Problem 2.3

$f(z) = z^3 - z^2 + 1$



By the Argument Principle,
f has 3 zeros inside Γ.

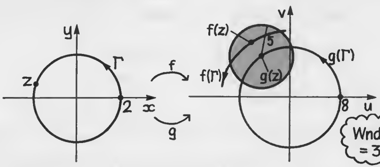
2. Why is Wnd $(f(\Gamma), 0) = 3$?

$f(z) = z^3 - z^2 + 1$

If $g(z) = z^3$, then $|g(z)| = 8$, for $z \in \Gamma$, and
$|f(z) - g(z)| = |-z^2 + 1| \le |z|^2 + 1 = 5$, for $z \in \Gamma$.

For $z \in \Gamma$,
$g(z)$ 'leads' $f(z)$.

z^3 is the dominant term



Wnd$(g(\Gamma),0) = 3$

Wnd $(f(\Gamma), 0) =$ Wnd$(g(\Gamma), 0) = 3$.

3. Rouché's Theorem

Suppose that:

1. f is analytic on a simply-connected region $\mathcal{R}$;
2. Γ is a simple-closed contour in $\mathcal{R}$;
3. g is analytic on $\mathcal{R}$ and
   $|f(z) - g(z)| < |g(z)|$, for $z \in \Gamma$.

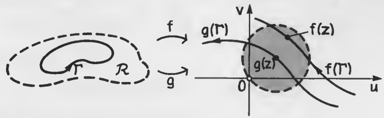
g(z) is the dominant term.

Then
f has the same number of zeros as g inside Γ.

counted according to their orders

4. Proof of Rouché's Theorem

Condition 3: If $z \in \Gamma$, then
$f(z)$ lies in the open disc with centre $g(z)$ and radius $|g(z)|$.



Since $g(z)$ 'leads' $f(z)$,
$f(\Gamma)$ winds round 0 as often as $g(\Gamma)$ does:
Wnd $(f(\Gamma), 0) =$ Wnd $(g(\Gamma), 0)$;
so f has the same number of zeros as g inside Γ.

Argument Principle

5. Zeros of $f(z) = e^z + 3z^2$ in $\{z : |z| < 1\}$

f is entire.

On $\Gamma = \{z : |z| = 1\}$ the dominant term is $g(z) = 3z^2$:

$$\begin{aligned} |f(z) - g(z)| &= |e^z| \\ &= e^{\operatorname{Re} z} \\ &\leq e^1 = e < 3 = |g(z)|, \quad \text{for } z \in \Gamma. \end{aligned}$$

Now g has a zero of order 2 at 0 (and no others), so, by Rouché's Theorem,

f has 2 zeros inside Γ.

6. Problem 2.4

(a) Determine how many zeros of the function
$$f(z) = z^5 + z^3 + iz + 1$$
lie in $\{z : |z| < 2\}$.

(b) Determine how many zeros of the function
$$f(z) = e^z - \tfrac{1}{3} z^4$$
lie in $\{z : |z| < 1\}$.

(c) Determine how many zeros of the function
$$f(z) = z^5 - 3z^3 - 1$$
lie in each of the following regions.

(i) $\{z : |z| < 2\}$ (ii) $\{z : |z| < 1\}$ (iii) $\{z : 1 < |z| < 2\}$

7. Fundamental Theorem of Algebra

'A non-constant polynomial function has at least one zero.'

Let
$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$
where $n \geq 1$, $a_n \neq 0$. Then p has exactly n zeros, all lying in $\{z : |z| < R\}$, where
$$R = 1 + \max\{|a_0|/|a_n|, \ldots, |a_{n-1}|/|a_n|\}.$$

Proof Put
$$f(z) = \frac{1}{a_n} p(z) = \frac{a_0}{a_n} + \frac{a_1}{a_n} z + \cdots + z^n,$$

f and p have the same zeros.

and $g(z) = z^n$. Then

$$\begin{aligned} |f(z) - g(z)| &= \left| \frac{a_0}{a_n} + \frac{a_1}{a_n} z + \cdots + \frac{a_{n-1}}{a_n} z^{n-1} \right| \\ &\leq \left| \frac{a_0}{a_n} \right| + \left| \frac{a_1}{a_n} \right| |z| + \cdots + \left| \frac{a_{n-1}}{a_n} \right| |z|^{n-1} \\ &\leq (R-1)(1 + |z| + \cdots + |z|^{n-1}) \\ &\leq (|z|-1)(1 + |z| + \cdots + |z|^{n-1}), \quad \text{for } |z| \geq R, \\ &= |z|^n - 1 < |g(z)|; \end{aligned}$$

Triangle Inequality

$$\Rightarrow |f(z) - g(z)| < |g(z)|, \quad \text{for } |z| \geq R. \qquad (*)$$

Now g has a zero of order n at 0 (and no others), so by Rouché's Theorem,

f has n zeros in $\{z : |z| < R\}$.

Finally,

f has no zeros outside $\{z : |z| < R\}$. ■

$f(z) = 0$ & $|z| \geq R$ contradicts (*).

Remark The statement of Rouché's Theorem is often formulated in slightly different ways to that in Frame 3. In one alternative formulation the key inequality

$$|f(z) - g(z)| < |g(z)|, \qquad \text{for } z \in \Gamma, \tag{2.4}$$

is replaced by

$$|f(z) - g(z)| < |f(z)|, \qquad \text{for } z \in \Gamma. \tag{2.5}$$

Since this amounts simply to exchanging the roles of f and g (because $|f(z) - g(z)| = |g(z) - f(z)|$), the conclusion once again is that f has the same number of zeros as g inside Γ.

In yet another formulation, the key inequality is replaced by

$$|g(z)| < |f(z)|, \qquad \text{for } z \in \Gamma.$$

Since this amounts to replacing g by $f + g$ in Inequality (2.5), the conclusion is now that f has the same number of zeros as $f + g$ inside Γ.

We chose the formulation (2.4) in order that the function g could play the role of a dominant term in f (rather as n^3 is a dominant term in $n^3 + 2n + 1$). Thus, given a function f whose zeros are to be investigated, the strategy is to split f into two functions g and $f - g$, such that

1. g dominates $f - g$ on Γ;
2. it is evident how many zeros g has inside Γ.

For example, in Frame 5 where $f(z) = e^z + 3z^2$ and $\Gamma = \{z : |z| = 1\}$, we chose $g(z) = 3z^2$.

(The choice of g is not unique; see the Solution to Exercise 2.3(a)(ii), for example.)

In the audio tape, the simple-closed contour Γ was always a circle, and the main problem was to choose a suitable dominant term g on Γ. In the following example, the choice of contour is also a problem.

Example 2.2

Prove that the equation

$$z + e^{-z} = 2 \tag{2.6}$$

has exactly one solution in the right half-plane $\{z : \operatorname{Re} z > 0\}$.

Solution

First note that Equation (2.6) can be rewritten in the form $f(z) = 0$, where

$$f(z) = z + e^{-z} - 2.$$

Next notice that the boundary of the right half-plane is not a simple-closed contour, so neither Rouché's Theorem nor the Argument Principle can be applied directly to this set. Instead, we consider open semi-discs of the form

Recall that ∂S denotes the boundary of S.

$$S = \{z : |z| < r, \operatorname{Re} z > 0\},$$

and prove that, for all sufficiently large r (actually, for $r > 3$), f has exactly one zero in S. It then follows that f has exactly one zero in $\{z : \operatorname{Re} z > 0\}$, as required.

We let $\Gamma = \partial S$, a simple-closed contour, and try to choose a dominant term on Γ. To do this it helps to notice that

$$|e^{-z}| = e^{-\operatorname{Re} z} \le 1, \qquad \text{for } z \in \Gamma, \tag{2.7}$$

since $\operatorname{Re} z \ge 0$, for $z \in \Gamma$. On the other hand, if $r > 3$, then

$$|z - 2| > 1, \qquad \text{for } z \in \Gamma, \tag{2.8}$$

since $\{z : |z - 2| = 1\}$ lies inside Γ when $r > 3$ (see Figure 2.5).

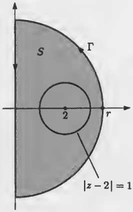


Figure 2.5

Combining Inequalities (2.7) and (2.8), we obtain

$$|e^{-z}| < |z-2|, \qquad \text{for } z \in \Gamma,$$

and so we take $g(z) = z - 2$ and $f(z) - g(z) = e^{-z}$. By Rouché's Theorem, we deduce that f has the same number of zeros as g inside Γ, namely one ($g(z) = 0 \iff z = 2$). Hence f has exactly one zero in $\{z : \operatorname{Re} z > 0\}$ as required. ■

Remark Notice also that

$$|e^{-z}| < |z-2|, \qquad \text{for } \operatorname{Re} z = 0,$$

so that f has exactly one zero in $\{z : \operatorname{Re} z \geq 0\}$.

Problem 2.5

Prove that, if $a > 1$, then the equation $z + e^{-z} = a$ has exactly one solution in $\{z : \operatorname{Re} z \geq 0\}$.

Problem 2.6

(a) Use the Taylor series about 0 for exp to prove that

$$|e^z - 1| \leq e - 1, \qquad \text{for } |z| \leq 1.$$

(b) Use the inequality in part (a) to prove that the equation

$$e^z = 2z + 1$$

has exactly one solution in $\{z : |z| < 1\}$.

We can often be more precise about the location of the zeros of a given analytic function f by studying the restriction of f to a particular set, such as the real axis. For example, in Frame 1 we saw that the function $f(z) = z^3 - z^2 + 1$ has three zeros inside $\Gamma = \{z : |z| = 2\}$. By sketching the graph of $x \longmapsto f(x)$ using calculus (see Figure 2.6), we see that f has exactly one real zero, which lies in the interval $]-1, 0[$, and so is inside Γ. It follows that the other two zeros of f must form a pair of complex conjugates (see *Unit A1*, Exercise 3.4). Indeed, if

$$f(z) = z^3 - z^2 + 1 = 0,$$

then

$$0 = \overline{z^3 - z^2 + 1} = \overline{z}^3 - \overline{z}^2 + 1 = f(\overline{z}).$$

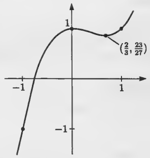


Figure 2.6 $f(x) = x^3 - x^2 + 1$

A similar argument can be applied whenever the function f satisfies

$$\overline{f(z)} = f(\overline{z}).$$

Recall from *Unit A2*, Exercise 4.4, that
$\overline{e^z} = e^{\overline{z}}$,
$\overline{\sin z} = \sin \overline{z}$.

Problem 2.7

Prove that the two zeros of the function $f(z) = e^z + 3z^2$ in $\{z : |z| < 1\}$ (see Frame 5) form a pair of complex conjugates.

Remarks

1 The same kind of argument cannot be used with a function such as

$$f(z) = z^5 + z^3 + iz + 1,$$

since the coefficients of this polynomial are not all real. In Problem 2.4(a) you found that the five zeros of this function lie inside $\{z : |z| < 2\}$ and it is natural to ask whether we can make a more precise statement about their location. For example, how many of these zeros lie in the upper half-plane? Since all five zeros lie inside $\{z : |z| < 2\}$, we need only find the number of zeros in the semi-disc $\{z : |z| < 2, \operatorname{Im} z > 0\}$. This could be determined from the Argument Principle. (In fact, the function f has three zeros in the upper half-plane.)

2 Locating zeros is important in many applications. For example, in 'control theory' it is a standard problem to decide whether a given polynomial function of the form

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}$, has *all* its zeros in the open left half-plane. Such real polynomials are called **stable**. Special algorithms for determining stability have been devised.

3 Later in the course we shall discuss how the Newton–Raphson method can be used to calculate approximations to the zeros of analytic functions.

Unit D3, Section 1

3 LOCAL BEHAVIOUR OF ANALYTIC FUNCTIONS

After working through this section, you should be able to:

(a) state the Open Mapping Theorem and appreciate its application to regions;

(b) state the Local Mapping Theorem and use it to determine the local behaviour of an analytic function;

(c) state and use the Inverse Function Rule;

(d) obtain the Taylor series of the inverse function of a given one-one analytic function.

3.1 The Open Mapping Theorem

We have seen that analytic functions have many remarkable properties and that many formulas hold for analytic functions which do not hold for general complex functions. Using the Argument Principle, we now demonstrate another property of analytic functions.

Theorem 3.1 Open Mapping Theorem

Let a function f be analytic and non-constant on a region $\mathcal{R}$ and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

The proof of Theorem 3.1 is given in Subsection 3.3.

Remark If f is a constant function, then $f(G)$ is a singleton set, which is certainly not open.

At first sight the Open Mapping Theorem may not seem particularly remarkable. However, many complex functions do not map open sets to open sets. For example, the function

$$f(z) = f(x+iy) = x^2 + iy^2$$

maps $\mathbb{C}$ to the first quadrant $\{u + iv : u \geq 0, v \geq 0\}$, which is not an open set (see Figure 3.1).

f is differentiable only at 0, by the Cauchy–Riemann Converse Theorem (*Unit A4*, Theorem 2.2).

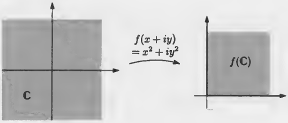


Figure 3.1

The Open Mapping Theorem provides an easy way of proving that there are no analytic functions with certain properties.

Problem 3.1

Prove that there are no non-constant entire functions which map $\mathbb{C}$ to $\mathbb{R}$.

One immediate consequence of the Open Mapping Theorem is that non-constant analytic functions map regions to regions.

Corollary If a function f is analytic and non-constant on a region $\mathcal{R}$, then $f(\mathcal{R})$ is also a region.

Proof A region $\mathcal{R}$ is a connected open set. The Open Mapping Theorem shows that $f(\mathcal{R})$ is open and Theorem 4.2 of *Unit A3* shows that $f(\mathcal{R})$ is connected. Hence $f(\mathcal{R})$ is also a region. ■

Next, we describe a result, closely related to the Open Mapping Theorem, which gives a description of the different types of *local* behaviour of an analytic function. First we look at a simple example.

Example 3.1

Let $f(z) = z^3$.

(a) Prove that, for each w such that $0 < |w| < 1$, the equation $f(z) = w$ has three distinct solutions in the open disc $\{z : 0 < |z| < 1\}$.

(b) Determine an open disc D with centre 1 such that the restriction of the function f to D is one-one.

(c) Why is it impossible to find an open disc D with centre 0 such that the restriction of f to D is one-one?

Solution

(a) If $0 < |w| < 1$, then w can be written in the form

$$w = \rho(\cos\phi + i\sin\phi), \quad 0 < \rho < 1, -\pi < \phi \le \pi.$$

See the solution to Exercise 1.6, *Unit A2*.

Then, by Theorem 3.1 of *Unit A1*, the equation $w = z^3$ has exactly three distinct solutions

$$z_k = \rho^{1/3}\left(\cos\left(\tfrac{1}{3}\phi + \tfrac{2}{3}k\pi\right) + i\sin\left(\tfrac{1}{3}\phi + \tfrac{2}{3}k\pi\right)\right), \quad k = 0, 1, 2, \tag{3.1}$$

which lie in $\{z : 0 < |z| < 1\}$ since $0 < \rho^{1/3} < 1$ (see Figure 3.2).

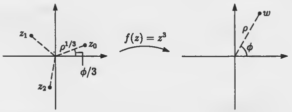


Figure 3.2

(b) Using Equation (3.1), we see that the restriction of f to the sector

$$A = \{z : |\operatorname{Arg} z| < \pi/3\}$$

is one-one (because at most one of the cube roots can lie in A).

Since $D = \{z : |z - 1| < \sqrt{3}/2\} \subseteq A$, because $\sin \pi/3 = \sqrt{3}/2$ (see Figure 3.3), it follows that the restriction of f to D is one-one, as required.

(c) Suppose that D is an open disc with centre 0. If $z \in D$ and $z \neq 0$, then

$$f\left(ze^{2\pi i/3}\right) = \left(ze^{2\pi i/3}\right)^3 = z^3 = f(z).$$

Since $ze^{2\pi i/3} \in D$, f is not one-one on D. ∎

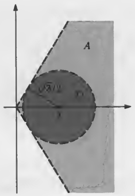


Figure 3.3

Problem 3.2

Let $f(z) = 1 + z^4$.

(a) Prove that for each w such that $0 < |w - 1| < \frac{1}{4}$ the equation $f(z) = w$ has four solutions in $\{z : 0 < |z| < 1/\sqrt{2}\}$.

(b) Determine an open disc D with centre $1 + i$ such that the restriction of f to D is one-one.

The functions $f(z) = z^3$ in Example 3.1 and $f(z) = 1 + z^4$ in Problem 3.2 illustrate various possible types of local behaviour of an analytic function. Although these functions are not one-one (on their domains), they are one-one *near* certain points. However, $f(z) = z^3$ is not one-one near $\alpha = 0$. Indeed a better description of the behaviour of $f(z) = z^3$ near 0 is *three-one*, since each point in $\{w : 0 < |w| < 1\}$ is the image of three points in $\{z : 0 < |z| < 1\}$. Similarly, the behaviour of $f(z) = 1 + z^4$ near $\alpha = 0$ may be described as *four-one*. These examples suggest the following definition, the form of which may seem rather strange; it is justified at the end of Remark 2.

> **Definition** Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then f is **n-one near α** if there is a region $\mathcal{S}$ in $\mathcal{R}$, with $\alpha \in \mathcal{S}$, and a function ϕ which is analytic and one-one on its domain $\mathcal{S}$, such that
>
> $$f(z) = f(\alpha) + (\phi(z))^n, \qquad \text{for } z \in \mathcal{S}. \qquad (3.2)$$

Remarks

1 Notice that if $n = 1$, then $\phi(z) = f(z) - f(\alpha)$, for $z \in \mathcal{S}$.

2 Equation (3.2) tells us that, on $\mathcal{S}$, f can be decomposed as the one-one function ϕ followed by the polynomial function $p(\zeta) = f(\alpha) + \zeta^n$; that is, $f = p \circ \phi$ on $\mathcal{S}$. This decomposition is illustrated in Figure 3.4. The region $\mathcal{S}$ around α (on the left) is mapped by ϕ to the region $\phi(\mathcal{S})$ around $\phi(\alpha) = 0$ (in the centre), and then any open disc $\Delta \subseteq \phi(\mathcal{S})$ centred at 0 is mapped by $p(\zeta) = f(\alpha) + \zeta^n$ to the open disc $p(\Delta)$ (on the right) centred at $f(\alpha)$.

We have used ζ, the Greek lower case letter zeta, in the specification of p since z is the domain variable for f and w is the codomain variable for f.

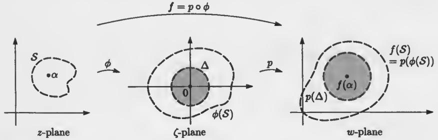


Figure 3.4

Each point in the disc $p(\Delta)$, apart from the centre $f(\alpha)$, is the image under p of exactly n points in $\phi(\mathcal{S})$, all lying in Δ, and hence the image under f of exactly n points in $\mathcal{S}$. This justifies the statement that f is n-one near α (and also shows that f can be n-one near α for at most one value of n).

Example 3.1(a) illustrates this definition with $f(z) = z^3$ and $n = 3$. Here $\alpha = 0$, $f(\alpha) = 0$, $\mathcal{S} = \{z : |z| < 1\}$ and $\phi(z) = z$ ($|z| < 1$). On the other hand, Example 3.1(b) illustrates this definition with $f(z) = z^3$ and $n = 1$. Here $\alpha = 1$, $f(\alpha) = 1$, $\mathcal{S} = \{z : |z - 1| < \sqrt{3}/2\}$ and $\phi(z) = f(z) - 1$ ($z \in \mathcal{S}$).

The following key result shows that the local behaviour of an analytic function f near a point α is determined by the order of the zero α of the function $z \longmapsto f(z) - f(\alpha)$. This, in turn, depends on the nature of the Taylor series for f about α.

Theorem 3.2 Local Mapping Theorem

Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z - \alpha)^n + a_{n+1}(z - \alpha)^{n+1} + \cdots, \qquad (3.3)$$

where $n \geq 1$ and $a_n \neq 0$, if and only if f is n-one near α.

The proof of Theorem 3.2 is given in Subsection 3.3.

Remarks

1 Theorem 3.2 tells us that the local behaviour of an analytic function f near a point α depends on the lowest-order non-vanishing derivative of f at α. Indeed the Taylor series for f about α takes the form in Equation (3.3) if and only if

$$0 = f'(\alpha) = f''(\alpha) = \ldots = f^{(n-1)}(\alpha), \text{ but } f^{(n)}(\alpha) \neq 0.$$

In particular, $f'(\alpha) \neq 0$ if and only if f is one-one near α.

2 Note that the value of $f(\alpha)$ has no bearing on the type of local behaviour of f at α.

3 The remark following the proof of Theorem 3.2 gives further information about the geometric behaviour of f when f is n-one near α.

In the next example we apply the Local Mapping Theorem (and the condition in Remark 1) to investigate the local behaviour of a given analytic function near several points of its domain.

Example 3.2

Describe the local behaviour of the function $f(z) = z^3 - z^2$ near each of the following points.

(a) $\alpha = 0$ (b) $\alpha = 1$ (c) $\alpha = \frac{2}{3}$

Solution

(a) Since the Taylor series about $\alpha = 0$ for f is

$$f(z) = -z^2 + z^3,$$

the function f is two-one near 0, by the Local Mapping Theorem.

(b) If $\alpha = 1$, then

$$f'(\alpha) = 3\alpha^2 - 2\alpha = 1 \neq 0,$$

so f is one-one near 1, by the Local Mapping Theorem.

(c) If $\alpha = \frac{2}{3}$, then

$$f'(\alpha) = 3\alpha^2 - 2\alpha = 0,$$
$$f''(\alpha) = 6\alpha - 2 = 2 \neq 0,$$

so f is two-one near $\frac{2}{3}$, by the Local Mapping Theorem. ∎

Problem 3.3

Let $f(z) = z - z^2$.

(a) Describe the local behaviour of the function f near each of the points $\alpha = 0$ and $\alpha = \frac{1}{2}$.

(b) Prove that the restriction of the function f to $\{z : |z| \leq \frac{1}{2}\}$ is one-one, but that this is false for any larger disc with centre 0.

(*Hint*: Show that if $z_1, z_2 \in \{z : |z| \leq \frac{1}{2}\}$ and $f(z_1) = f(z_2)$, then $z_1 = z_2$.)

3.2 Inverse functions

We are now going to use the Local Mapping Theorem to obtain an improved version of the Inverse Function Rule, first discussed in *Unit A4*, Section 3. In that unit we showed that if $f : A \longrightarrow B$ is a one-one function whose inverse function f^{-1} is continuous at $\beta \in B$, and if f is differentiable at $f^{-1}(\beta)$ with $f'(f^{-1}(\beta)) \neq 0$, then f^{-1} is differentiable at β with

$$(f^{-1})'(\beta) = \frac{1}{f'(f^{-1}(\beta))}.$$

We can now give a much stronger version of this result for analytic functions.

Theorem 3.3 Inverse Function Rule

Let f be a one-one analytic function whose domain is a region $\mathcal{R}$. Then f^{-1} is analytic on $f(\mathcal{R})$ and

$$(f^{-1})'(\beta) = \frac{1}{f'(f^{-1}(\beta))}, \qquad \text{for } \beta \in f(\mathcal{R}). \qquad (3.4)$$

Proof To prove the result we need to show that

(a) $f'(\alpha) \neq 0$, for $\alpha \in \mathcal{R}$, and

(b) f^{-1} is continuous on $f(\mathcal{R})$.

Then we can apply the version of the Inverse Function Rule in *Unit A4*, to deduce that f^{-1} is differentiable at each $\beta \in f(\mathcal{R})$ and that Equation (3.4) holds.

To prove (a) note that if $f'(\alpha) = 0$ for some $\alpha \in \mathcal{R}$ then, by the Local Mapping Theorem, f is n-one near α, for some $n > 1$; this contradicts the hypothesis that f is one-one on $\mathcal{R}$.

To prove (b) let $\beta \in f(\mathcal{R})$ and put $\alpha = f^{-1}(\beta)$. We want to show that for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$|w - \beta| < \delta \implies |f^{-1}(w) - \alpha| < \varepsilon \qquad (3.5)$$

(see Figure 3.5).

This is the ε-δ definition of continuity, applied to f^{-1} at β.

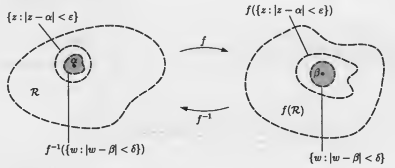


Figure 3.5

However, we know that $f(\{z : |z - \alpha| < \varepsilon\})$ is an open set, by the Open Mapping Theorem, and so there exists $\delta > 0$ such that

$$\{w : |w - \beta| < \delta\} \subseteq f(\{z : |z - \alpha| < \varepsilon\}$$

(see the right-hand side of Figure 3.5), which implies that

$$f^{-1}(\{w : |w - \beta| < \delta\}) \subseteq \{z : |z - \alpha| < \varepsilon\}.$$

Hence Implication (3.5) holds, as required. ■

In *Unit B3*, Section 4, we introduced the inverse tangent function $\tan^{-1}$ and the inverse sine function $\sin^{-1}$, and stated some of their properties. In particular, the function $\tan^{-1}$ is the inverse function of the restriction of tan to the strip $\{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$, whereas $\sin^{-1}$ is the inverse function of the restriction of sin to this strip. The domains of $\tan^{-1}$ and $\sin^{-1}$ are shown in Figure 3.6.

We justify these domains in *Unit D1*.

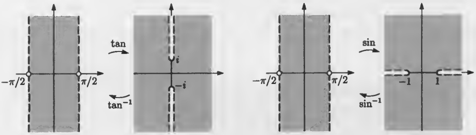


Figure 3.6

Theorem 3.3 can be used to show that these inverse functions are analytic.

Example 3.3

Prove that the function

$$f(z) = \tan z \qquad (-\pi/2 < \operatorname{Re} z < \pi/2)$$

has an inverse function f^{-1} which is analytic.

Solution

To show that f is one-one on the region $\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$, note that if $z_1, z_2 \in \mathcal{R}$, then

Here we are using the first strategy in *Unit A2*, Subsection 1.5 to show that f has an inverse function.

$$\begin{aligned}
\tan z_1 = \tan z_2 \quad &\Longrightarrow \quad \frac{\sin z_1}{\cos z_1} = \frac{\sin z_2}{\cos z_2} \\
&\Longrightarrow \quad \sin z_1 \cos z_2 = \sin z_2 \cos z_1 \\
&\Longrightarrow \quad \sin(z_1 - z_2) = 0 \\
&\Longrightarrow \quad z_1 - z_2 = n\pi, \quad \text{where } n \in \mathbb{Z}, \\
&\Longrightarrow \quad z_1 = z_2,
\end{aligned}$$

Unit A2, Theorem 4.2

because $|\operatorname{Re}(z_1 - z_2)| < \pi$. Since f is analytic on $\mathcal{R}$, we deduce, by the Inverse Function Rule, that f^{-1} is analytic on $f(\mathcal{R})$. ■

Further, it follows from Equation (3.4) that, for $\beta \in f(\mathcal{R})$,

$$(f^{-1})'(\beta) = (\tan^{-1})'(\beta) = \frac{1}{1+\beta^2},$$

as for the real inverse tan function.

Problem 3.4

Prove that each of the following functions has an analytic inverse function.

(a) $f(z) = \sin z \quad (-\pi/2 < \operatorname{Re} z < \pi/2)$

(b) $f(z) = z - z^2 \quad (|z| < \frac{1}{2})$

(*Hint for part (b)*: See Problem 3.3.)

Problem 3.5

Let a function f be analytic at a point α, and suppose that $f'(\alpha) \neq 0$. Prove that there is a region S, with $\alpha \in S$, such that the restriction of f to S has an analytic inverse function.

Theorem 3.3 guarantees the existence of an analytic inverse function f^{-1} for any one-one analytic function f. It is natural then to try and determine the Taylor series for this inverse function about a point β in the image of f in terms of the Taylor series for the original function about $\alpha = f^{-1}(\beta)$. The next example illustrates one method of doing this.

Example 3.4

Use the Taylor series about 0 for the function

$$f(z) = \sin z \qquad (-\pi/2 < \operatorname{Re} z < \pi/2)$$

to determine the first three non-zero terms of the Taylor series about 0 for the function $f^{-1}(w) = \sin^{-1} w$.

In *Unit B3*, we obtained this Taylor series by using the Integration Rule.

Solution

First note that

$$f^{-1}(f(z)) = \sin^{-1}(\sin z) = z. \tag{3.6}$$

Thus the Taylor series about 0 for $f^{-1} \circ f$ reduces to the single power z^1, and the coefficients of the other powers vanish.

Now the Taylor series for $f^{-1} \circ f$ can be found, alternatively, by using the Composition Rule for Taylor series (*Unit B3*, Frame 10). We know that

$$f(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots, \qquad \text{for } |z| < \pi/2,$$

and we may assume that the Taylor series about 0 for f^{-1} takes the form

Unit B3, Problem 3.4

$$f^{-1}(w) = b_1 w + b_3 w^3 + b_5 w^5 + \cdots,$$

since f^{-1} is an odd function (because f is). Thus, by Equation (3.6) and the Composition Rule for Taylor series,

If $f^{-1}(w) = z$, then
$$f(z) = w \Longrightarrow f(-z) = -w,$$
so
$$f^{-1}(-w) = -z.$$

$$z = b_1\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) + b_3\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^3 + b_5\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^5 + \cdots.$$

Equating the coefficients of $z, z^3, z^5, \ldots$, we obtain a sequence of equations for the coefficients $b_1, b_3, b_5, \ldots$:

$$\begin{aligned}
z&: \quad 1 = b_1 && \Longrightarrow b_1 = 1\\
z^3&: \quad 0 = -\frac{1}{3!}b_1 + b_3 && \Longrightarrow b_3 = \frac{1}{6}b_1 = \frac{1}{6}\\
z^5&: \quad 0 = \frac{1}{5!}b_1 - \frac{3}{3!}b_3 + b_5 && \Longrightarrow b_5 = \frac{1}{2}b_3 - \frac{1}{5!}b_1 = \frac{3}{40}.
\end{aligned}$$

The coefficient of
$$z^2\left(-\frac{z^3}{3!}\right)$$
in
$$\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)^3$$
is 3.

Hence the required Taylor series is

$$\sin^{-1} w = w + \frac{1}{6}w^3 + \frac{3}{40}w^5 + \cdots. \quad ■$$

Example 3.4 was fairly straightforward because both the Taylor series for f and f^{-1} were about 0. In general, we know the Taylor series for f about a point α:

$$f(z) = a_0 + a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots,$$

and wish to find the Taylor series for f^{-1} about the point $\beta = f(\alpha)$:

$$f^{-1}(w) = b_0 + b_1(w-\beta) + b_2(w-\beta)^2 + \cdots.$$

Since $\beta = f(\alpha) = a_0$ and $\alpha = f^{-1}(\beta) = b_0$, these two Taylor series can be written in the form

$$f(z) - \beta = a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots$$

and

$$f^{-1}(w) - \alpha = b_1(w-\beta) + b_2(w-\beta)^2 + \cdots.$$

Hence the identity

$$\begin{aligned} z - \alpha &= f^{-1}(f(z)) - \alpha \\ &= b_1\left(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\right) \\ &\quad + b_2\left(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\right)^2 + \cdots \end{aligned}$$

can be used to obtain $b_1, b_2, \ldots$, in terms of $a_1, a_2, \ldots$. The method, known informally as 'inverting a Taylor series' is summarized in the following strategy.

Strategy for inverting a Taylor series

Given the Taylor series about α for f:

$$f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n,$$

where $a_1 = f'(\alpha) \neq 0$, we can find the Taylor series about $\beta = f(\alpha)$ for f^{-1}:

$$f^{-1}(w) = \sum_{n=0}^{\infty} b_n(w-\beta)^n,$$

by putting $b_0 = \alpha$ and equating the powers of $(z-\alpha)$ in the identity

$$\begin{aligned} z - \alpha &= b_1\left(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\right) \\ &\quad + b_2\left(a_1(z-\alpha) + a_2(z-\alpha)^2 + \cdots\right)^2 + \cdots, \end{aligned}$$

to obtain equations for $b_1, b_2, \ldots$, in terms of $a_1, a_2, \ldots$.

Remember that these Taylor series representations about α and β are valid on suitable open discs, with centres α and β respectively.

Remark The assumption that $f'(\alpha) \neq 0$ is needed in order to guarantee that f is one-one near α (so that the restriction of f to a region $\mathcal{S}$ about α has an analytic inverse function).

Problem 3.6

Use the above strategy to invert the Taylor series about $\alpha = 0$ for each of the following functions, giving the first three non-vanishing terms in each case.

(a) $f(z) = e^z$

(b) $f(z) = z - z^2$

3.3 Proofs

We conclude this section by proving the Open Mapping Theorem and the Local Mapping Theorem.

This subsection may be omitted on a first reading.

> **Theorem 3.1 Open Mapping Theorem**
>
> Let a function f be analytic and non-constant on a region $\mathcal{R}$ and let G be an open subset of $\mathcal{R}$. Then $f(G)$ is open.

Proof To prove that $f(G)$ is open, we need to show that if $\beta \in f(G)$, then there exists $\varepsilon > 0$ such that

$$\{w : |w - \beta| < \varepsilon\} \subseteq f(G).$$

Since $\beta \in f(G)$, we know that there exists $\alpha \in G$ such that $f(\alpha) = \beta$. Furthermore, the solutions of the equation $f(z) = \beta$ are isolated (since $z \longrightarrow f(z) - \beta$ is analytic), and so we can choose an open disc D in G with centre α and radius sufficiently small that $f(z) \neq \beta$, for $z \in D - \{\alpha\}$. Thus, if C is a circle in D with centre α, then the image $f(C)$ is a closed contour which does not pass through β (see Figure 3.7).

Unit B3, Theorem 5.3

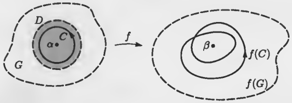


Figure 3.7

Since $f(C)$ is compact, being the continuous image of a compact set, the complement of $f(C)$ is open. Hence we can choose $\varepsilon > 0$ so that $\{w : |w - \beta| < \varepsilon\}$ lies in the complement of $f(C)$ (see Figure 3.8).

Unit A3, Theorem 5.4

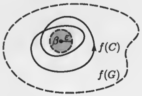


Figure 3.8

By Theorem 1.5, the winding number of $f(C)$ round each point of the disc $\{w : |w - \beta| < \varepsilon\}$ is equal to $\mathrm{Wnd}(f(C), \beta)$. Now

$$\mathrm{Wnd}(f(C), \beta) \geq 1, \tag{3.7}$$

by the corollary to the Argument Principle, since $f(\alpha) = \beta$. Hence

$$\mathrm{Wnd}(f(C), w) \geq 1, \qquad \text{for } |w - \beta| < \varepsilon.$$

Since $f(\alpha) = \beta$, the equation $f(z) = \beta$ has at least one solution inside C.

(This justifies the position of β in Figures 3.7 and 3.8.) Thus, by the corollary to the Argument Principle again, the equation $f(z) = w$ has at least one solution z inside C, for each w such that $|w - \beta| < \varepsilon$. Hence $\{w : |w - \beta| < \varepsilon\} \subseteq f(G)$, as required. ■

Theorem 3.2 Local Mapping Theorem

Let a function f be analytic on a region $\mathcal{R}$ and let $\alpha \in \mathcal{R}$. Then the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z-\alpha)^n + a_{n+1}(z-\alpha)^{n+1} + \cdots,$$

where $n \geq 1$ and $a_n \neq 0$, if and only if f is n-one near α.

Proof Because f is n-one near α for at most one value of n, it is sufficient to prove that if the Taylor series about α for f has the above form, then f is n-one near α. Now

$$\begin{aligned} f(z) - f(\alpha) &= a_n(z-\alpha)^n + a_{n+1}(z-\alpha)^{n+1} + \cdots \\ &= (z-\alpha)^n(a_n + a_{n+1}(z-\alpha) + \cdots) \\ &= (z-\alpha)^n g(z), \end{aligned} \tag{3.8}$$

say, where the function g is analytic at α with $g(\alpha) = a_n \neq 0$. Thus the function $f(z) - f(\alpha)$ has a zero of order n at α. Equation (3.8) shows that the result is at least believable since, if z is near α, then

$$f(z) - f(\alpha) \cong (z-\alpha)^n g(\alpha).$$

Now assume that $n = 1$, so that $f'(\alpha) = a_1 \neq 0$, let $\beta = f(\alpha)$ and choose the open disc D, the circle C and radius $\varepsilon > 0$ as in the proof of Theorem 3.1. Equation (3.7) now takes the more precise form

$$\operatorname{Wnd}(f(C), \beta) = 1,$$

since $f(z) - f(\alpha)$ has a simple zero at α. Hence, by Theorem 1.5,

$$\operatorname{Wnd}(f(C), w) = 1, \qquad \text{for } |w - \beta| < \varepsilon. \tag{3.9}$$

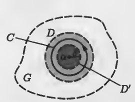


Figure 3.9

Because f is continuous at α, we can choose $\delta > 0$ so small that $D' = \{z : |z-\alpha| < \delta\}$ lies inside C (see Figure 3.9) and

$$|z - \alpha| < \delta \implies |f(z) - \beta| < \varepsilon.$$

Hence, by Equation (3.9), the restriction of f to D' is one-one, as required.

Next suppose that $n > 1$. Since $g(\alpha) = a_n \neq 0$, we can choose a logarithm function, Log_ϕ say, such that the function

$$h(z) = \exp\left(\frac{1}{n}\operatorname{Log}_\phi(g(z))\right)$$

is analytic on an open disc D'', say, with centre α, and we then have

$$\begin{aligned} (h(z))^n &= \left(\exp\left(\frac{1}{n}\operatorname{Log}_\phi(g(z))\right)\right)^n \\ &= g(z), \qquad \text{for } z \in D''. \end{aligned}$$

Therefore Equation (3.8) gives

$$\begin{aligned} f(z) - f(\alpha) &= (z-\alpha)^n h(z)^n \\ &= (\phi(z))^n, \qquad \text{for } z \in D'', \end{aligned}$$

where ϕ is the analytic function

$$\phi(z) = (z-\alpha)h(z) \qquad (z \in D'').$$

The distinction between the use of ϕ as an analytic function and of ϕ in Log_ϕ should be clear from the context.

Thus we can write

$$f(z) = f(\alpha) + (\phi(z))^n = p(\phi(z)), \qquad \text{for } z \in D'',$$

where p is the polynomial function

$$p(\zeta) = f(\alpha) + \zeta^n \qquad (\zeta \in \mathbb{C}).$$

Now, $\phi(\alpha) = 0$ and $\phi'(\alpha) = h(\alpha) \neq 0$. Thus we can apply the $n = 1$ case to ϕ to obtain a disc D''', say, with centre α such that the restriction of ϕ to D''' is one-one. Hence f is indeed n-one near α. ■

Remark The proof of the Local Mapping Theorem actually gives rather precise information about the behaviour of f near α. By the Inverse Function Rule, the function ϕ^{-1} is analytic on $\phi(D''')$, and so ϕ^{-1} maps an open disc Δ with centre 0 onto a region $\mathcal{S}$, say, which contains α (see Figure 3.10). Thus $f(\mathcal{S}) = p(\phi(\mathcal{S}))$ is an open disc with centre $f(\alpha)$ and the radius of $f(\mathcal{S})$ parallel to the real axis (shown in Figure 3.10) is the image of n evenly-spaced radii in Δ. These are in turn the images of n smooth paths emerging from α in $\mathcal{S}$. Because ϕ^{-1} is conformal at 0 (by the Inverse Function Rule), these n smooth paths in $\mathcal{S}$ have evenly-spaced tangent vectors at α (see the left-hand side of Figure 3.10). Hence $\mathcal{S}$ is divided into n more or less equal 'sectors', each of which is mapped by f onto $f(\mathcal{S})$.

ϕ^{-1} is conformal by *Unit A4*, Theorem 4.2 with $f = \phi^{-1}$.

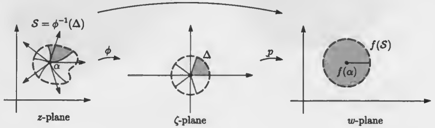


Figure 3.10

Moreover, if Γ_1 and Γ_2 are *any* two smooth paths emerging from α, and θ is the angle from Γ_1 to Γ_2, then the paths $f(\Gamma_1)$ and $f(\Gamma_2)$ both emerge from $f(\alpha)$ and the angle from $f(\Gamma_1)$ to $f(\Gamma_2)$ is $n\theta$ (see Figure 3.11).

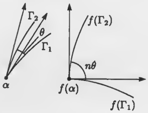


Figure 3.11

4 EXTREME VALUES OF ANALYTIC FUNCTIONS

After working through this section, you should be able to:

(a) appreciate how the Open Mapping Theorem leads to the Local Maximum Principle;

(b) understand the Maximum Principle and apply it to find the maximum modulus of an analytic function on a compact set, in appropriate cases;

(c) understand the Minimum Principle, the Boundary Uniqueness Theorem and Schwarz's Lemma and apply them in appropriate cases.

4.1 The Local Maximum Principle

In Section 3 we showed that non-constant analytic functions map regions to regions. This property has an important application to the extrema (that is, the maxima and minima) of the modulus $|f|$ of an analytic function f. We begin by defining the notion of a *local maximum*.

The function $|f|$ has the same domain as f and rule

$$z \longmapsto |f(z)|.$$

> **Definition** Let a function f be defined on a region $\mathcal{R}$. Then the function $|f|$ has a **local maximum** at the point $\alpha \in \mathcal{R}$ if there is some $r > 0$ such that $\{z : |z - \alpha| < r\} \subseteq \mathcal{R}$ and
>
> $$|f(z)| \le |f(\alpha)|, \qquad \text{for } |z - \alpha| < r.$$

For example, if $f(z) = \exp(-|z|^2)$, then $|f| = f$ has a local maximum at 0 (see Figure 4.1). However, the graphs of the surfaces $s = |f(z)|$ for

$$f(z) = z^2 \quad \text{and} \quad f(z) = e^{-z},$$

shown in Figures 4.2 and 4.3, suggest that the corresponding functions $|f|$ have no local maxima at all. (In each case, the part of the domain shown is $\{z = x + iy : -2 \le x \le 2, -2 \le y \le 2\}$.)

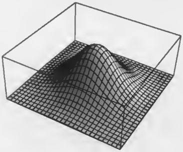


Figure 4.1
$s = \exp(-|z|^2) = e^{-x^2-y^2}$

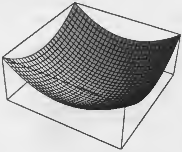


Figure 4.2 $s = |z^2| = x^2 + y^2$

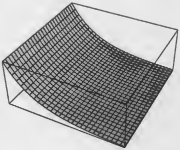


Figure 4.3 $s = |e^{-z}| = e^{-x}$

The explanation for this absence of local maxima is as follows. Suppose that f is a non-constant analytic function whose domain is a region $\mathcal{R}$, and that $\alpha \in \mathcal{R}$. If $|f|$ does have a local maximum at α, then there is some $r > 0$ such that

$$|f(z)| \le |f(\alpha)|, \qquad \text{for } |z - \alpha| < r. \tag{4.1}$$

But if $D = \{z : |z - \alpha| < r\}$, then $f(D)$ is an open set containing $f(\alpha)$ (by the Open Mapping Theorem) and so $f(D)$ contains an open disc, D' say, centred at $f(\alpha)$ (see Figure 4.4).

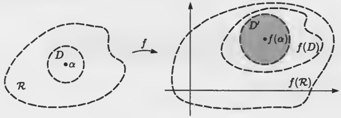


Figure 4.4

We now claim that there is some point w in D' such that

$$|w| > |f(\alpha)|.$$

If $f(\alpha) = 0$, then this is evident because f is non-constant. If $f(\alpha) \ne 0$, then such a w can be found by extending the line segment from 0 to $f(\alpha)$ slightly (see Figure 4.5).

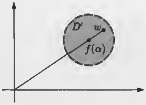


Figure 4.5

Since $w \in D'$, we have $w = f(z)$, for some $z \in D$, and hence

$$|f(z)| > |f(\alpha)|,$$

which contradicts Inequality (4.1). Thus we have proved the following result.

> **Theorem 4.1 Local Maximum Principle**
>
> Let a function f be analytic on a region $\mathcal{R}$. If f is non-constant on $\mathcal{R}$, then the function $|f|$ has no local maxima on $\mathcal{R}$.

Remark This result shows yet again how different the behaviour of complex analytic functions is from standard real functions. For example, the complex function $f(z) = \cos z$ is non-constant and entire and so by the Local Maximum Principle the function $z \longmapsto |\cos z|$ has no local maxima on $\mathbb{C}$. But the real function $x \longmapsto |\cos x|$ has infinitely many local maxima (see Figure 4.6, which illustrates these facts for $-2\pi \le x \le 2\pi, 0 \le y \le 2.5$).

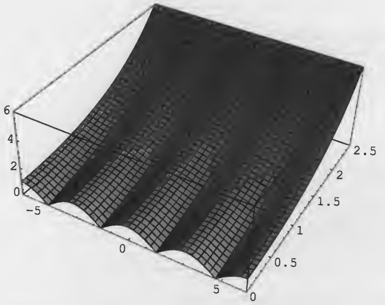


Figure 4.6 $s = |\cos z|$

Figures 4.1, 4.2, 4.3 and 4.6 were produced using the software package *Mathematica* (Wolfram Research Inc).

The Local Maximum Principle often simplifies the process of finding the maximum modulus of an analytic function on a given set. Suppose, for example, that we want to determine

$$\max\{|z^2 - 4z - 3| : |z| \le 2\}.$$

The function

$$f(z) = z^2 - 4z - 3$$

is continuous on $\mathbb{C}$ and $\{z : |z| \le 2\}$ is a compact set, so this maximum certainly exists by the Extreme Value Theorem. Since f is also analytic on the open disc $D = \{z : |z| < 2\}$ and non-constant there, we deduce by the Local Maximum Principle that f has no local maximum on D. Hence the maximum value of $|f|$ on $\{z : |z| \le 2\}$ must be taken on the boundary $\{z : |z| = 2\}$, that is,

Unit A3, Theorem 5.2

$$\max\{|z^2 - 4z - 3| : |z| \le 2\} = \max\{|z^2 - 4z - 3| : |z| = 2\}.$$

Thus we have reduced the problem of finding the maximum of $|f|$ on the set $\{z : |z| \le 2\}$ to that of finding its maximum on $\{z : |z| = 2\}$, a considerable simplification (although still not easy!).

In the next subsection we formulate a general result of this type and then return to the above problem (in Example 4.1).

4.2 The Maximum Principle

First we recall from *Unit A3*, Section 5 various notions about sets illustrated in Figure 4.7.

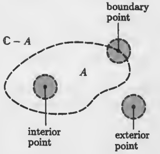


Figure 4.7

> **Definitions** Let A be a subset of $\mathbb{C}$ and let $\alpha \in \mathbb{C}$. Then
>
> (a) α is an **interior point** of A if there is an open disc centred at α which lies entirely in A;
>
> (b) α is an **exterior point** of A if there is an open disc centred at α which lies entirely outside A;
>
> (c) α is a **boundary point** of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

These three types of points form the **interior** of A, written $\operatorname{int} A$, the **exterior** of A, written $\operatorname{ext} A$, and the **boundary** of A, written ∂A.

It is evident that the sets $\operatorname{int} A$, ∂A and $\operatorname{ext} A$ are always disjoint and that

$$\operatorname{int} A \cup \partial A \cup \operatorname{ext} A = \mathbb{C}.$$

Thus

$$\operatorname{int} A \cup \partial A = \mathbb{C} - \operatorname{ext} A$$

is always closed (since $\operatorname{ext} A$ is open), and is called the *closure* of A.

Unit A3, Theorem 5.5

> **Definition** The **closure** $\overline{A}$ of a set A in $\mathbb{C}$ is
>
> $$\overline{A} = \operatorname{int} A \cup \partial A.$$

For our purposes, the notation $\overline{\mathcal{R}}$, where as usual $\mathcal{R}$ is a region, is a handy shorthand for the union of $\mathcal{R}$ with its boundary $\partial\mathcal{R}$. For example, if $\mathcal{R} = \{z : |z| < 1\}$, then $\partial R = \{z : |z| = 1\}$ and so $\overline{\mathcal{R}} = \{z : |z| \leq 1\}$.

> **Theorem 4.2 Maximum Principle**
>
> Let a function f be analytic on a bounded region $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that
>
> $$|f(z)| \leq |f(\alpha)|, \qquad \text{for } z \in \overline{\mathcal{R}}. \tag{4.2}$$

This result is sometimes called the Maximum Modulus Theorem.

Proof First we note that $\overline{\mathcal{R}}$ is closed. Also, $\mathcal{R}$ is bounded and so

$$\mathcal{R} \subseteq \{z : |z| \leq M\},$$

for some $M > 0$. Hence

$$\partial\mathcal{R} \subseteq \{z : |z| \leq M\},$$

since all points outside $\{z : |z| \leq M\}$ are exterior to $\mathcal{R}$, and so

$$\overline{\mathcal{R}} \subseteq \{z : |z| \leq M\}.$$

Thus $\overline{\mathcal{R}}$ is a compact set. By the Extreme Value Theorem, therefore, as f is continuous on $\overline{\mathcal{R}}$ there exists $\alpha \in \overline{\mathcal{R}}$ such that

$$|f(z)| \leq |f(\alpha)|, \qquad \text{for } z \in \overline{\mathcal{R}}.$$

If $\alpha \in \partial\mathcal{R}$, then the proof is complete. Otherwise $\alpha \in \mathcal{R}$, and so f must be constant on $\mathcal{R}$, by the Local Maximum Principle.

It then follows, by the continuity of f on $\overline{\mathcal{R}}$, that f is constant on $\overline{\mathcal{R}}$ also. Indeed, if β is any point of $\partial\mathcal{R}$, then there exists a sequence $\{z_n\}$ in $\mathcal{R}$ such that $z_n \to \beta$ (choose z_n in $\mathcal{R} \cap \{z : |z - \beta| < 1/n\}$, as in Figure 4.8), and hence $f(z_n) \to f(\beta)$, by the continuity of f at β. Thus if $f(z) = \lambda$, say, for $z \in \mathcal{R}$, then $f(z_n) = \lambda$, for $n = 1, 2, \ldots$, so that $f(\beta) = \lambda$ also.

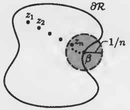


Figure 4.8

Hence $|f|$ is constant on $\overline{\mathcal{R}}$ and so Inequality (4.2) holds for any point α of ∂R. ■

Remark Note that if the function f is non-constant and analytic on $\mathcal{R}$, then Inequality (4.2) can be strengthened to

$$|f(z)| < |f(\alpha)|, \qquad \text{for } z \in \mathcal{R}.$$

Indeed, if $|f(z)| = |f(\alpha)|$ for some $z \in \mathcal{R}$, then f must have a local maximum at z, which is not possible by the Local Maximum Principle.

Now we return to the example discussed at the end of Subsection 4.1.

Example 4.1

Show that

$$\max\{|z^2 - 4z - 3| : |z| \le 2\} = 7\sqrt{7/3},$$

and state where this maximum value is attained.

In this example, we obtain the best possible estimate discussed at the end of *Unit A1*, Section 5.

Solution

Since $f(z) = z^2 - 4z - 3$ is analytic on the open disc $D = \{z : |z| < 2\}$ and continuous on $\overline{D} = \{z : |z| \le 2\}$, it follows from the Maximum Principle that there exists $\alpha \in \partial D = \{z : |z| = 2\}$ such that

$$\max\{|f(z)| : |z| \le 2\} = |f(\alpha)|.$$

Since each point of ∂D has the form $2e^{it}$, for some $t \in [0, 2\pi]$, we need to determine

$$\max\{|f(2e^{it})| : 0 \le t \le 2\pi\}.$$

Recall that

$$\gamma(t) = 2e^{it} \quad (t \in [0, 2\pi])$$

is the standard parametrization of $\{z : |z| = 2\}$.

Because

$$\begin{aligned} f(2e^{it}) &= (2e^{it})^2 - 4(2e^{it}) - 3 \\ &= 4e^{i2t} - 8e^{it} - 3 \\ &= (4\cos 2t - 8\cos t - 3) + i(4\sin 2t - 8\sin t), \end{aligned}$$

we obtain

$$\begin{aligned} |f(2e^{it})|^2 &= (4\cos 2t - 8\cos t - 3)^2 + (4\sin 2t - 8\sin t)^2 \\ &= \left(16\cos^2 2t + 64\cos^2 t + 9 - 64\cos 2t\cos t - 24\cos 2t + 48\cos t\right) \\ &\quad + \left(16\sin^2 2t + 64\sin^2 t - 64\sin 2t\sin t\right) \\ &= 16 + 64 + 9 - 64(\cos 2t\cos t + \sin 2t\sin t) + 48\cos t - 24\cos 2t \\ &= 89 - 64\cos(2t - t) + 48\cos t - 24\left(2\cos^2 t - 1\right) \\ &= 113 - 16\cos t - 48\cos^2 t. \end{aligned}$$

Alternatively, use $|f(2e^{it})|^2 = f(2e^{it})\overline{f(2e^{it})}$.

This expression for $|f(2e^{it})|^2$ is a quadratic in $\cos t$ and so, by completing the square, we obtain

$$\begin{aligned} |f(2e^{it})|^2 &= 48\left(\frac{113}{48} - \frac{1}{3}\cos t - \cos^2 t\right) \\ &= 48\left(\frac{113}{48} + \frac{1}{36} - \left(\frac{1}{6} + \cos t\right)^2\right) \\ &= \frac{343}{3} - 48\left(\frac{1}{6} + \cos t\right)^2. \end{aligned}$$

The maximum of this expression is 343/3, obtained when $\frac{1}{6} + \cos t = 0$, that is, when $\cos t = -\frac{1}{6}$, which corresponds to

$$\alpha = 2e^{it} = 2\left(-\tfrac{1}{6} \pm i\sqrt{1 - \left(\tfrac{1}{6}\right)^2}\right)$$
$$= \tfrac{1}{3}\left(-1 \pm i\sqrt{35}\right),$$

shown in Figure 4.9. Thus

$$\max\left\{\left|f\left(2e^{it}\right)\right| : 0 \le t \le 2\pi\right\} = \sqrt{343/3} = 7\sqrt{7/3},$$

and so, by the Maximum Principle,

$$\max\left\{\left|z^2 - 4z - 3\right| : |z| \le 2\right\} = 7\sqrt{7/3},$$

as required. ■

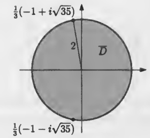


Figure 4.9

Problem 4.1

Determine each of the following maxima.

(a) $\max\left\{\left|z^2 - 1\right| : |z| \le 1\right\}$

(b) $\max\left\{\left|z^2 - 2\right| : |z - 1| \le 1\right\}$

(c) $\max\left\{\left|z^3 - 1\right| : 0 \le \operatorname{Re} z \le 1, 0 \le \operatorname{Im} z \le 1\right\}$

(*Hint*: In part (c) you will need to consider the four sides of the square separately.)

We conclude this section by stating a number of corollaries to the Maximum Principle.

> **Corollary 1 Minimum Principle**
>
> Let a function f be analytic on a bounded region $\mathcal{R}$, and continuous and non-zero on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that
>
> $$|f(z)| \ge |f(\alpha)|, \qquad \text{for } z \in \overline{\mathcal{R}}.$$

Remark Notice the assumption here that f is non-zero on $\overline{\mathcal{R}}$. Without this, the result would be false. For example, if $f(z) = z$ and $\mathcal{R} = \{z : |z| < 1\}$, then $|f(z)| = 1$, for $z \in \partial\mathcal{R}$, but

$$\min\{|f(z)| : |z| \le 1\} = |f(0)| = 0.$$

In general, if $f(\alpha) = 0$ for some $\alpha \in \mathcal{R}$, then

$$\min\{|f(z)| : z \in \mathcal{R}\} = |f(\alpha)| = 0.$$

Problem 4.2

(a) Prove the Minimum Principle by applying the Maximum Principle to the function $g = 1/f$.

(b) Determine

$$\min\left\{\left|\exp\left(z^2\right)\right| : |z| \le 1\right\}.$$

The next corollary is related to Cauchy's Integral Formula

$$f(\alpha) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - \alpha}\,dz,$$

which expresses the value of an analytic function f inside a simple-closed contour Γ in terms of the values of f on Γ. Thus the value of f at each point inside Γ is determined by the values of f on Γ. This last assertion holds in the

following more general situation (where Cauchy's Integral Formula does not apply).

Corollary 2 Boundary Uniqueness Theorem

Let functions f and g be analytic on a bounded region $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. If $f = g$ on $\partial\mathcal{R}$, then $f = g$ on $\mathcal{R}$.

Problem 4.3

Prove the Boundary Uniqueness Theorem by applying the Maximum Principle to the function $h = g - f$.

Our final corollary plays an important role in showing that certain analytic functions must take a particular form (see, for example, Problem 4.4).

Corollary 3 Schwarz's Lemma

Hermann Schwarz (1843–1921) was a pupil of Weierstrass.

Let a function f be analytic on $\{z : |z| < R\}$ with $f(0) = 0$, and suppose that

$$|f(z)| \le M, \qquad \text{for } |z| < R.$$

Then

$$|f(z)| \le (M/R)|z|, \qquad \text{for } |z| < R.$$

Remark Schwarz's Lemma has the following simple geometric interpretation: if the analytic function f maps the open disc $\{z : |z| < R\}$ into the open disc $\{z : |z| < M\}$ with $f(0) = 0$, then the image of any point z on the circle Γ with equation $|z| = r$ lies inside or on the circle $|w| = M(r/R)$. (See Figure 4.10.)

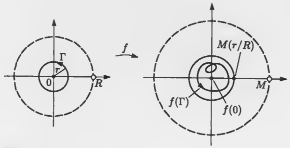


Figure 4.10

Proof Consider the Taylor series about 0 for f, which takes the form

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots, \qquad \text{for } |z| < R,$$

(since $f(0) = 0$). Then

$$\frac{f(z)}{z} = a_1 + a_2 z + a_3 z^2 + \cdots, \qquad \text{for } 0 < |z| < R.$$

Thus the function

$$g(z) = a_1 + a_2 z + a_3 z^2 + \cdots \qquad (|z| < R)$$

provides an analytic extension of $z \longmapsto f(z)/z$ to $\{z : |z| < R\}$.

We now apply the Maximum Principle to g on the open disc $\{z : |z| < r\}$, where $0 < r < R$ (we cannot take $r = R$ here, since f (and hence g) is not known to be continuous on $\{z : |z| \le R\}$).

This gives

$$\begin{aligned} |g(z)| \le |g(\alpha)| &= \frac{|f(\alpha)|}{|\alpha|}, \qquad \text{where } |\alpha| = r, \\ &\le \frac{M}{r}, \qquad \text{for } |z| \le r. \end{aligned} \tag{4.3}$$

Since Inequality (4.3) holds for all r such that $|z| \le r < R$, we deduce, on letting r tend to R, that

$$|g(z)| \le \frac{M}{R}, \qquad \text{for } |z| < R.$$

Hence

$$\left|\frac{f(z)}{z}\right| \le \frac{M}{R}, \qquad \text{for } 0 < |z| < R,$$

so that

$$|f(z)| \le \left(\frac{M}{R}\right)|z|, \qquad \text{for } 0 < |z| < R.$$

Since this inequality evidently holds for $z = 0$ as well, the proof of Schwarz's Lemma is complete. ■

We end this section by asking you to solve a problem which shows the power of Schwarz's Lemma.

Problem 4.4

Suppose that f is a one-one analytic function with domain $D = \{z : |z| < 1\}$ such that

$$f(0) = 0 \qquad \text{and} \qquad f(D) = D.$$

Deduce that f is of the form $f(z) = \lambda z$, where $|\lambda| = 1$; that is, f is a rotation.

(*Hint*: Apply Schwarz's Lemma to both f and f^{-1} to show that $|f(z)| = |z|$, for $|z| < 1$, and then prove that the function $g(z) = f(z)/z$ is constant.)

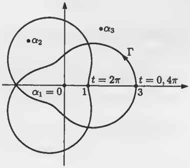
EXERCISES

Section 1

Exercise 1.1 The closed path

$$\Gamma : \gamma(t) = \left(2 + \cos \tfrac{3}{2}t\right) e^{it} \qquad (t \in [0, 4\pi])$$

is shown in the figure in the margin.



(a) Determine by inspection the winding number of Γ round each of the following points: $\alpha_1 = 0$, α_2 and α_3.

(b) Determine a continuous argument function for Γ and hence verify the value of $\mathrm{Wnd}(\Gamma, 0)$ that you found in part (a).

Exercise 1.2 Determine a continuous argument function for the path

$$\Gamma : \gamma(t) = -1 + it \qquad (t \in [-1, 1])$$

and hence evaluate $\mathrm{Wnd}(\Gamma, 0)$.
(*Hint*: Note that $\Gamma \subseteq \mathbb{C}_{2\pi}$.)

Exercise 1.3 Evaluate each of the following expressions.

(a) $\mathrm{Log}_{3\pi}(-i)$ (b) $\mathrm{Log}_{2\pi}(2)$ (c) $\mathrm{Log}_{3\pi/2}(-3)$

Exercise 1.4 Evaluate

$$\int_\Gamma \frac{1}{z-2}\, dz,$$

where Γ is the (smooth) path in Exercise 1.1.

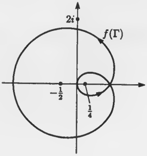
Section 2

Exercise 2.1

(a) Let $f(z) = (z-1)^3(z-2)^2(z-3)$. Determine the poles of the function f'/f and find the residues of f'/f.

(b) Let $\Gamma = \{z : |z| = 4\}$ and let f be the function in part (a). Determine $\mathrm{Wnd}(f(\Gamma), 0)$.

Exercise 2.2 Let $f(z) = z - z^2$ and $\Gamma = \{z : |z| = 1\}$. Use the diagram of the image path $f(\Gamma)$ in the margin to determine the number of solutions inside Γ for each of the following equations.



(a) $f(z) = -\frac{1}{2}$ (b) $f(z) = \frac{1}{4}$ (c) $f(z) = 2i$

Exercise 2.3 For each of the following functions f, determine the number of zeros of f in the given sets.

(a) $f(z) = z^5 + 3z + 10$;

(i) $S_1 = \{z : |z| < 2\}$, (ii) $S_2 = \{z : |z| < 1\}$,

(iii) $S_3 = \{z : 1 < |z| < 2\}$, (iv) $S_4 = \{z : \mathrm{Im}\, z > 0\}$.

(b) $f(z) = 3z + \mathrm{Log}(1+z)$; $S = \left\{z : |z| < \frac{1}{2}\right\}$

(*Hint*: In part (b) use the estimate

$$|\mathrm{Log}(1+z)| \le 2|z|, \qquad \text{for } |z| \le \tfrac{1}{2},$$

which may be proved by using the Taylor series about 0 for $z \longmapsto \mathrm{Log}(1+z)$.)

Section 3

Exercise 3.1 Prove that if a function f is analytic on the open unit disc $\{z : |z| < 1\}$ and

$$|f(z)| = \pi, \qquad \text{for } |z| < 1,$$

then f is constant.

Exercise 3.2 Let $f(z) = z - z^2$ and let $\alpha = \frac{1}{2}$. Determine a function ϕ such that

$$f(z) = f(\alpha) + (\phi(z))^2, \qquad \text{for } z \in \mathbb{C},$$

and hence show that the function f satisfies the definition of two-one near α.

Exercise 3.3 Use the Local Mapping Theorem to describe the local behaviour of each of the following functions near the given points α.

(a) $f(z) = \cos z$, $\alpha = 0$

(b) $f(z) = e^z$, $\alpha = 2\pi i$

(c) $f(z) = \sin z - z$, $\alpha = 0$

Exercise 3.4 Invert the Taylor series for the following functions, giving the first three non-vanishing terms in each case.

(a) $f(z) = z^3 + 3z$ about $\alpha = 0$

(b) $f(z) = e^z$ about $\alpha = 1$

Section 4

Exercise 4.1 Determine each of the following maxima, stating at which points they are attained.

(a) $\max\{|z^2 + 2| : |z| \leq 1\}$

(b) $\max\{|z^2 - 2| : |z - i| \leq 1\}$

(c) $\max\{|e^{z^2}| : -1 \leq \operatorname{Re} z \leq 1, -1 \leq \operatorname{Im} z \leq 1\}$

(d) $\max\{|\tan z| : -\pi/4 \leq \operatorname{Re} z \leq \pi/4, -1 \leq \operatorname{Im} z \leq\}$

(*Hint*: You will find Example 4.4 and Problem 4.7 of *Unit A2* useful in part (d).)

Exercise 4.2 Under the assumptions of Schwarz's Lemma, show that

(a) $|f'(0)| \leq M/R$;

(b) if $|f(z_0)| = (M/R)|z_0|$, for some $|z_0| < R$, or if $|f'(0)| = M/R$, then

$$f(z) = \lambda z \qquad (|z| < R),$$

where λ is a constant with $|\lambda| = M/R$.

SOLUTIONS TO THE PROBLEMS

Section 1

1.1 **(a)** Once $(2 \times (+1) + (-1))$.

(b) None.

1.2 **(a)** Since $|\gamma(t)| = t$, for $t \in [1,3]$, we have

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{\pi i t}, \quad \text{for } t \in [1,3].$$

Thus one choice of continuous argument function is

$$\theta(t) = \pi t \quad (t \in [1,3]).$$

(b) Since $\gamma(t) = \left(1+t^2\right)^{1/2} e^{i \tan^{-1} t}$, for $t \in [0,1]$, we have

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i \tan^{-1} t}, \quad \text{for } t \in [0,1].$$

Thus one choice of continuous argument function is

$$\theta(t) = \tan^{-1} t \quad (t \in [0,1]).$$

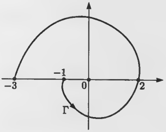
Remark Note that $\tan^{-1} t = \operatorname{Arg}(\gamma(t))$.

(c) Here it is easy to see that one choice of continuous argument function is

$$\theta(t) = -4t \quad (t \in [0, 2\pi]).$$

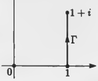
1.3 **(a)**

$$\begin{aligned}\operatorname{Wnd}(\Gamma, 0) &= \frac{1}{2\pi}(\theta(3) - \theta(1)) \\ &= \frac{1}{2\pi}(3\pi - \pi) = 1.\end{aligned}$$



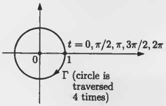
(b)

$$\begin{aligned}\operatorname{Wnd}(\Gamma, 0) &= \frac{1}{2\pi}(\theta(1) - \theta(0)) \\ &= \frac{1}{2\pi}(\tan^{-1} 1 - \tan^{-1} 0) = \frac{1}{8}.\end{aligned}$$



(c)

$$\begin{aligned}\operatorname{Wnd}(\Gamma, 0) &= \frac{1}{2\pi}(\theta(2\pi) - \theta(0)) \\ &= \frac{1}{2\pi}(-4 \times 2\pi - 0) = -4.\end{aligned}$$



1.4 **(a)** Let one continuous argument function for Γ be

$$\theta : t \longmapsto \theta(t) \quad (t \in [a,b]).$$

Then

$$\theta : t \longmapsto \theta(t) \quad (t \in [a,c])$$

is a continuous argument function for Γ_1 and

$$\theta : t \longmapsto \theta(t) \quad (t \in [c,b])$$

is a continuous argument function for Γ_2.

Now

$$\operatorname{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

$$\operatorname{Wnd}(\Gamma_1, 0) = \frac{1}{2\pi}(\theta(c) - \theta(a))$$

and

$$\operatorname{Wnd}(\Gamma_2, 0) = \frac{1}{2\pi}(\theta(b) - \theta(c)).$$

Hence

$$\begin{aligned}\operatorname{Wnd}(\Gamma_1, 0) + \operatorname{Wnd}(\Gamma_2, 0) &= \frac{1}{2\pi}(\theta(c) - \theta(a)) \\ &\quad + \frac{1}{2\pi}(\theta(b) - \theta(c)) \\ &= \frac{1}{2\pi}(\theta(b) - \theta(a)) \\ &= \operatorname{Wnd}(\Gamma, 0),\end{aligned}$$

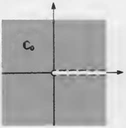
as required.

1.5 **(a)** $\operatorname{Arg}_\pi(i) = \pi/2$, since $\pi/2$ is the argument of i which lies in $]-\pi, \pi]$.

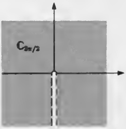
(b) $\operatorname{Arg}_0(-1) = -\pi$, since $-\pi$ is the argument of -1 which lies in $]-2\pi, 0]$.

(c) $\operatorname{Arg}_{3\pi/2}(1-i) = -\pi/4$, since $-\pi/4$ is the argument of $1-i$ which lies in $]-\pi/2, 3\pi/2]$.

1.6 (a)



(b)



1.7 $\mathrm{Wnd}(\Gamma, 1) = 2$, $\mathrm{Wnd}(\Gamma, -2) = 1$, $\mathrm{Wnd}(\Gamma, -2i) = 1$, $\mathrm{Wnd}(\Gamma, 3i) = 0$.

1.8
$$\begin{aligned}
\mathrm{Wnd}(\Gamma, \alpha) &= \mathrm{Wnd}(\Gamma - \alpha, 0) \quad \text{(by Equation (1.4))}\\
&= \frac{1}{2\pi i}\int_{\Gamma-\alpha} \frac{1}{z}\,dz \quad \text{(by Theorem 1.4)}\\
&= \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)-\alpha}\,dt \quad \text{(by Equation (1.3))}\\
&= \frac{1}{2\pi i}\int_\Gamma \frac{1}{z-\alpha}\,dz,
\end{aligned}$$
as required.

Section 2

2.1 Since $f(z) = z^{10}(z-1)$,
$$f'(z) = 10z^9(z-1) + z^{10}$$
and so
$$\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{10z^9(z-1)+z^{10}}{z^{10}(z-1)}\\
&= \frac{10}{z} + \frac{1}{z-1}.
\end{aligned}$$
Thus f'/f has a simple pole at 1 with residue 1, whereas f has a simple zero at 1. Also f'/f has a simple pole at 0 with residue 10, whereas f has a zero at 0 of order ten.

2.2 From Figure 2.3 we can see that $\mathrm{Wnd}(f(\Gamma), -1) = 3$. Hence the equation $f(z) = -1$ has exactly three solutions inside Γ.

2.3 Since $\mathrm{Wnd}(f(\Gamma), 0) = 3$, f has three zeros in $\{z : |z| < 2\}$.

2.4 (a) On $\Gamma = \{z : |z| = 2\}$ a dominant term is $g(z) = z^5$, since
$$\begin{aligned}
|f(z) - g(z)| &= |z^3 + iz + 1|\\
&\le |z|^3 + |z| + 1 \quad \text{(Triangle Inequality)}\\
&= 11 < 32 = |g(z)|, \quad \text{for } z \in \Gamma.
\end{aligned}$$
Now g has a zero of order five at 0, which is in $\{z : |z| < 2\}$ (and no other zeros); so, by Rouché's Theorem, f has five zeros in $\{z : |z| < 2\}$.

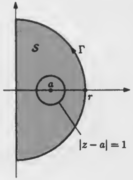
(b) On $\Gamma = \{z : |z| = 1\}$ a dominant term is $g(z) = e^z$, since
$$\begin{aligned}
|f(z) - g(z)| &= |-\tfrac{1}{3}z^4|\\
&= \tfrac{1}{3}, \quad \text{for } z \in \Gamma,
\end{aligned}$$
and
$$\begin{aligned}
|g(z)| &= |e^z| = e^{\mathrm{Re}\, z}\\
&\ge e^{-1} > \tfrac{1}{3}, \quad \text{for } z \in \Gamma,
\end{aligned}$$
so that
$$|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma.$$
Now g has no zeros in $\{z : |z| < 1\}$, and so, by Rouché's Theorem, f has no zeros in $\{z : |z| < 1\}$.

(c) (i) On $\Gamma = \{z : |z| = 2\}$ a dominant term is $g(z) = z^5$, since
$$\begin{aligned}
|f(z) - g(z)| &= |-3z^3 - 1|\\
&\le 3|z|^3 + 1 \quad \text{(Triangle Inequality)}\\
&= 25 < 32 = |g(z)|, \quad \text{for } z \in \Gamma.
\end{aligned}$$
Now g has a zero of order five at 0, which is in $\{z : |z| < 2\}$ (and no other zeros); so, by Rouché's Theorem, f has five zeros in $\{z : |z| < 2\}$.

(ii) On $\Gamma = \{z : |z| = 1\}$ a dominant term is $g(z) = -3z^3$, since
$$\begin{aligned}
|f(z) - g(z)| &= |z^5 - 1|\\
&\le |z|^5 + 1 \quad \text{(Triangle Inequality)}\\
&= 2 < 3 = |g(z)|, \quad \text{for } z \in \Gamma. \qquad (1)
\end{aligned}$$
Now g has a zero of order three at 0, which is in $\{z : |z| < 1\}$ (and no other zeros); so, by Rouché's Theorem, f has three zeros in $\{z : |z| < 1\}$.

(iii) It follows from parts (i) and (ii) that f has two $(= 5 - 3)$ zeros in $\{z : 1 < |z| < 2\}$. (Note that f has no zeros on $\{z : |z| = 1\}$ in view of Inequality (1).)

2.5 We follow the argument in Example 2.2. Consider the function
$$f(z) = z + e^{-z} - a, \quad \text{where } a > 1,$$
and the semi-disc $S = \{z : |z| < r, \mathrm{Re}\, z > 0\}$. Let $\Gamma = \partial S$.



If $r > a+1$, a dominant term on Γ is $g(z) = z - a$, since

$$\begin{aligned}|f(z) - g(z)| &= |e^{-z}| = e^{-\operatorname{Re} z}\\ &\le 1, \quad \text{for } z \in \Gamma,\end{aligned}$$

and

$$\begin{aligned}|g(z)| &= |z-a|\\ &> 1, \quad \text{for } z \in \Gamma,\end{aligned}$$

since $\{z : |z-a| = 1\}$ lies inside Γ.

Hence $|f(z) - g(z)| < |g(z)|$, for $z \in \Gamma$ (with $r > a+1$). Since g has just one zero, a simple one at a, inside Γ, it follows from Rouché's Theorem that f has exactly one zero inside Γ.

Also, $|e^{-z}| < |z-a|$, for $\operatorname{Re} z = 0$, and so f has exactly one zero in $\{z : \operatorname{Re} z \ge 0\}$. Thus the equation $z + e^{-z} = a$ has exactly one solution in $\{z : \operatorname{Re} z \ge 0\}$.

2.6 (a) The Taylor series about 0 for exp is

$$\exp z = e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Hence

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad \text{for } z \in \mathbb{C},$$

and so

$$\begin{aligned}|e^z - 1| &= \left|z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right|\\ &\le |z| + \frac{|z|^2}{2!} + \frac{|z|^3}{3!} + \cdots \\ &\qquad (\textit{Unit B3}, \text{ Theorem 1.8})\\ &\le 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots, \quad \text{for } |z| \le 1,\\ &= e^1 - 1 = e - 1.\end{aligned}$$

Thus

$$|e^z - 1| \le e - 1, \quad \text{for } |z| \le 1.$$

(b) Let $\Gamma = \{z : |z| = 1\}$ and let f be the function

$$f(z) = e^z - 2z - 1.$$

On Γ a dominant term is $g(z) = -2z$, since

$$\begin{aligned}|f(z) - g(z)| &= |e^z - 1|\\ &\le e - 1, \quad \text{for } z \in \Gamma \text{ (by part (a))},\\ &< 2 = |g(z)|, \quad \text{for } z \in \Gamma.\end{aligned}$$

Now g has just one zero, a simple one at 0, inside Γ, so by Rouché's Theorem f has exactly one zero inside Γ. Hence the equation $e^z = 2z + 1$ has exactly one solution in $\{z : |z| < 1\}$.

2.7 The function $x \longmapsto e^x + 3x^2$ is positive for all $x \in \mathbb{R}$, and so the function $f(z) = e^z + 3z^2$ has no real zeros. Also

$$\begin{aligned}\overline{f(z)} &= \overline{e^z + 3z^2}\\ &= \overline{e^z} + \overline{3z^2} \quad (\textit{Unit A1}, \text{ Theorem 1.1(b)})\\ &= e^{\overline{z}} + 3\overline{z}^2 \quad (\textit{Unit A2}, \text{ Exercise 4.4(a) and } \textit{Unit A1}, \text{ Theorem 1.1(b)})\\ &= f(\overline{z}).\end{aligned}$$

Hence, the two zeros of the function f in $\{z : |z| < 1\}$ are complex conjugates.

Section 3

3.1 Let f be a non-constant entire function which maps $\mathbb{C}$ to $\mathbb{R}$. Then, by Theorem 3.1, $f(\mathbb{C})$ is open. But $f(\mathbb{C}) \subseteq \mathbb{R}$, and so $f(\mathbb{C}) = \varnothing$ (no open disc with centre $x \in f(\mathbb{C})$ lies entirely in $f(\mathbb{C})$).

Since $f(\mathbb{C}) \ne \varnothing$, we deduce that no such function f exists.

3.2 (a) If $0 < |w - 1| < \frac{1}{4}$, then $w - 1$ can be written in the form

$$w - 1 = \rho(\cos\phi + i\sin\phi), \ 0 < \rho < \tfrac{1}{4}, \ -\pi < \phi \le \pi.$$

Then, by Theorem 3.1 of *Unit A1*, the equation $1 + z^4 = w$ has exactly four distinct solutions

$$z_k = \rho^{1/4}\left(\cos\left(\tfrac{1}{4}\phi + \tfrac{1}{2}k\pi\right) + i\sin\left(\tfrac{1}{4}\phi + \tfrac{1}{2}k\pi\right)\right), \quad k = 0,1,2,3,$$

which lie in $\{z : 0 < |z| < 1/\sqrt{2}\}$ since $0 < \rho^{1/4} < 1/\sqrt{2}$.

(b) It follows that the restriction of f to the sector

$$A = \{z : 0 < \operatorname{Arg} z < \pi/2\}$$

is one-one (because at most one of the fourth roots can lie in A). Since

$$D = \{z : |z - 1 - i| < 1\} \subseteq A,$$

it follows that the restriction of f to D is one-one, as required.

3.3 (a) If $\alpha = 0$, then

$$f'(\alpha) = 1 - 2\alpha = 1 \ne 0,$$

so the function f is one-one near 0, by the Local Mapping Theorem.

If $\alpha = \frac{1}{2}$, then

$$f'(\alpha) = 1 - 2\alpha = 0,$$
$$f''(\alpha) = -2 \ne 0,$$

so f is two-one near $\frac{1}{2}$, by the Local Mapping Theorem.

(b) If z_1, z_2 both lie in $D = \left\{z : |z| \le \frac{1}{2}\right\}$, then

$$\begin{aligned}f(z_1) = f(z_2) &\Longrightarrow z_1 - z_1^2 = z_2 - z_2^2\\ &\Longrightarrow z_1 - z_2 = z_1^2 - z_2^2\\ &\Longrightarrow z_1 - z_2 = (z_1 - z_2)(z_1 + z_2)\\ &\Longrightarrow z_1 = z_2 \text{ or } 1 = z_1 + z_2.\end{aligned}$$

Now $1 = z_1 + z_2$ can happen for $z_1, z_2 \in D$ only if $z_1 = z_2 = \frac{1}{2}$ (since $\operatorname{Re}(z_1 + z_2) = \operatorname{Re} z_1 + \operatorname{Re} z_2 < 1$ otherwise). Hence the restriction of $f(z) = z - z^2$ to D is one-one.

If $r > \frac{1}{2}$, then the restriction of f to $\{z : |z| < r\}$ cannot be one-one, since f is two-one near $\frac{1}{2}$, by part (a).

3.4 (a) To prove that $f(z) = \sin z$ is one-one on $\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ note that if $z_1, z_2 \in \mathcal{R}$, then

$$\begin{aligned}\sin z_1 = \sin z_2 &\Longrightarrow \frac{1}{2i}\left(e^{iz_1} - e^{-iz_1}\right) = \frac{1}{2i}\left(e^{iz_2} - e^{-iz_2}\right)\\ &\Longrightarrow e^{iz_1} - e^{iz_2} = e^{-iz_1} - e^{-iz_2}\\ &\Longrightarrow e^{iz_1} - e^{iz_2} = \frac{e^{iz_2} - e^{iz_1}}{e^{i(z_1+z_2)}}\\ &\Longrightarrow e^{iz_1} = e^{iz_2} \text{ or } e^{i(z_1+z_2)} = -1.\end{aligned}$$

If $e^{iz_1} = e^{iz_2}$, then $iz_1 = iz_2 + 2n\pi i$, for some $n \in \mathbb{Z}$, so that $z_1 = z_2$, since $|\operatorname{Re}(z_1 - z_2)| < \pi$.

If $e^{i(z_1+z_2)} = -1$, then $i(z_1 + z_2) = i\pi + 2n\pi i$, for some $n \in \mathbb{Z}$, which is impossible since $|\operatorname{Re}(z_1 + z_2)| < \pi$.

Hence, for $z_1, z_2 \in \mathcal{R}$,

$$\sin z_1 = \sin z_2 \Longrightarrow z_1 = z_2,$$

as required.

Since f is evidently analytic on $\mathcal{R}$, we deduce, by the Inverse Function Rule, that f^{-1} is analytic on $f(\mathcal{R})$.

(b) It follows from Problem 3.3(b) that f is one-one on $\{z : |z| < \frac{1}{2}\}$. Since f is analytic on $\mathcal{R}$, we deduce, by the Inverse Function Rule, that f^{-1} is analytic on $f(\mathcal{R})$.

3.5 If f is analytic at α with $f'(\alpha) \neq 0$, then f is one-one near α, by the Local Mapping Theorem. This means that there is a region S, with $\alpha \in S$, such that the restriction of f to S is one-one, and so this restriction has an analytic inverse function, by the Inverse Function Rule.

3.6 **(a)** The Taylor series about $\alpha = 0$ for $f(z) = e^z$ is

$$f(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots.$$

Thus the Taylor series for f^{-1} about $\beta = f(\alpha) = 1$ is

$$f^{-1}(w) = b_0 + b_1(w-1) + b_2(w-1)^2 + b_3(w-1)^3 + \cdots,$$

where $b_0 = \alpha = 0$ and $b_1, b_2, b_3, \ldots$ satisfy

$$z = b_1\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right) + b_2\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)^2 + b_3\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right)^3 + \cdots.$$

Equating coefficients of powers of z, we obtain:

$$\begin{aligned}
z&: \quad 1 = b_1 &&\Longrightarrow b_1 = 1\\
z^2&: \quad 0 = \frac{1}{2!}b_1 + b_2 &&\Longrightarrow b_2 = -\tfrac{1}{2}b_1 = -\tfrac{1}{2}\\
z^3&: \quad 0 = \frac{1}{3!}b_1 + \frac{2}{2!}b_2 + b_3 &&\Longrightarrow b_3 = -\tfrac{1}{6}b_1 - b_2 = \tfrac{1}{3}.
\end{aligned}$$

Hence

$$f^{-1}(w) = (w-1) - \frac{(w-1)^2}{2} + \frac{(w-1)^3}{3} - \cdots.$$

This is the Taylor series about 1 for $\operatorname{Log} w$, as expected.

(b) The Taylor series about $\alpha = 0$ for $f(z) = z - z^2$ is

$$f(z) = z - z^2.$$

Thus the Taylor series about $\beta = f(\alpha) = 0$ for f^{-1} is

$$f^{-1}(w) = b_0 + b_1 w + b_2 w^2 + b_3 w^3 + \cdots,$$

where $b_0 = \alpha = 0$ and $b_1, b_2, b_3, \ldots$ satisfy

$$z = b_1(z - z^2) + b_2(z - z^2)^2 + b_3(z - z^2)^3 + \cdots.$$

Equating coefficients of powers of z, we obtain:

$$\begin{aligned}
z&: \quad 1 = b_1 &&\Longrightarrow b_1 = 1\\
z^2&: \quad 0 = -b_1 + b_2 &&\Longrightarrow b_2 = b_1 = 1\\
z^3&: \quad 0 = -2b_2 + b_3 &&\Longrightarrow b_3 = 2b_2 = 2.
\end{aligned}$$

Hence

$$f^{-1}(w) = w + w^2 + 2w^3 + \cdots.$$

Section 4

4.1 **(a)** Since $f(z) = z^2 - 1$ is analytic on $\{z : |z| < 1\}$ and continuous on $\{z : |z| \leq 1\}$, it follows from the Maximum Principle that there exists $\alpha \in \partial D = \{z : |z| = 1\}$ such that

$$\max\{|f(z)| : |z| \leq 1\} = |f(\alpha)|.$$

Since each point of ∂D has the form e^{it}, for $t \in [0, 2\pi]$, we need to determine

$$\max\{|f(e^{it})| : 0 \leq t \leq 2\pi\}.$$

Because

$$\begin{aligned}
f(e^{it}) &= e^{i2t} - 1\\
&= \cos 2t - 1 + i \sin 2t,
\end{aligned}$$

we obtain

$$\begin{aligned}
|f(e^{it})|^2 &= (\cos 2t - 1)^2 + \sin^2 2t\\
&= 2 - 2\cos 2t.
\end{aligned}$$

Now, the maximum of $2 - 2\cos 2t$ on $[0, 2\pi]$ is 4, obtained when $\cos 2t = -1$ (that is, when $t = \pi/2, 3\pi/2$). Thus

$$\max\{|f(e^{it})| : 0 \leq t \leq 2\pi\} = \sqrt{4} = 2,$$

and so

$$\max\{|z^2 - 1| : |z| \leq 1\} = 2.$$

Remark This maximum could have been found by simply noting that

$$\begin{aligned}
|z^2 - 1| &\leq |z|^2 + 1 \quad \text{(Triangle Inequality)}\\
&\leq 1 + 1 = 2,
\end{aligned}$$

for $|z| \leq 1$, and that equality is obtained by taking $z = \pm i$.

(b) Since $f(z) = z^2 - 2$ is analytic on $\{z : |z - 1| < 1\}$ and continuous on $\{z : |z - 1| \leq 1\}$, it follows from the Maximum Principle that there exists $\alpha \in \partial D = \{z : |z - 1| = 1\}$ such that

$$\max\{|f(z)| : |z - 1| \leq 1\} = |f(\alpha)|.$$

Since each point of ∂D has the form $1 + e^{it}$, for $t \in [0, 2\pi]$, we need to determine

$$\max\{|f(1 + e^{it})| : 0 \leq t \leq 2\pi\}.$$

Because

$$\begin{aligned}
f(1 + e^{it}) &= (1 + e^{it})^2 - 2\\
&= (\cos 2t + 2\cos t - 1) + i(\sin 2t + 2\sin t),
\end{aligned}$$

we obtain

$$\begin{aligned}
|f(1 + e^{it})|^2 &= (\cos 2t + 2\cos t - 1)^2 + (\sin 2t + 2\sin t)^2\\
&= (\cos^2 2t + 4\cos^2 t + 1 + 4\cos 2t \cos t\\
&\qquad - 2\cos 2t - 4\cos t)\\
&\quad + (\sin^2 2t + 4\sin^2 t + 4\sin 2t \sin t)\\
&= 6 + 4\cos(2t - t) - 4\cos t - 2\cos 2t\\
&= 6 - 2\cos 2t.
\end{aligned}$$

Now, the maximum of $6 - 2\cos 2t$ on $[0, 2\pi]$ is 8, obtained when $\cos 2t = -1$ (that is, when $t = \pi/2, 3\pi/2$). Thus

$$\max\{|f(1 + e^{it})| : 0 \leq t \leq 2\pi\} = \sqrt{8} = 2\sqrt{2},$$

and so

$$\max\{|z^2 - 2| : |z - 1| \leq 1\} = 2\sqrt{2}.$$

(c) Since $f(z) = z^3 - 1$ is analytic on the open square

$$S = \{z : 0 < \operatorname{Re} z < 1, 0 < \operatorname{Im} z < 1\}$$

and continuous on $\overline{S}$, it follows from the Maximum Principle that the required maximum is attained for some point of ∂S.

In order to study the four sides of the square S we note that if $z = x + iy$, then

$$\begin{aligned} z^3 - 1 &= (x+iy)^3 - 1 \\ &= \left(x^3 - 3xy^2 - 1\right) + i\left(3x^2y - y^3\right), \end{aligned}$$

so that

$$\begin{aligned} |z^3 - 1|^2 &= \left(x^3 - 3xy^2 - 1\right)^2 + \left(3x^2y - y^3\right)^2 \\ &= x^6 + y^6 + 3x^2y^2\left(x^2 + y^2\right) + 6xy^2 - 2x^3 + 1. \end{aligned}$$

There are four sides to consider.

Side 1: $y = 0, 0 \le x \le 1$

In this case

$$\begin{aligned} |z^3 - 1|^2 &= x^6 - 2x^3 + 1 \\ &= \left(x^3 - 1\right)^2, \end{aligned}$$

which attains a maximum value (on $[0,1]$) of 1 at $x = 0$.

Side 2: $x = 0, 0 \le y \le 1$

In this case

$$|z^3 - 1|^2 = y^6 + 1,$$

which attains a maximum value (on $[0,1]$) of 2 at $y = 1$.

Side 3: $x = 1, 0 \le y \le 1$

In this case

$$|z^3 - 1|^2 = y^6 + 3y^4 + 9y^2,$$

which attains a maximum value (on $[0,1]$) of 13 at $y = 1$.

Side 4: $y = 1, 0 \le x \le 1$

In this case

$$|z^3 - 1|^2 = x^6 + 3x^4 - 2x^3 + 3x^2 + 6x + 2.$$

Now the function

$$\phi(x) = x^6 + 3x^4 - 2x^3 + 3x^2 + 6x + 2$$

attains its maximum value of 13 on $[0,1]$ at 1, since

$$\begin{aligned} \phi'(x) &= 6x^5 + 12x^3 - 6x^2 + 6x + 6 \\ &\ge 0, \quad \text{for } 0 \le x \le 1, \end{aligned}$$

because

$$6x \ge 6x^2, \quad \text{for } 0 \le x \le 1.$$

Thus

$$\max\{|f(z)| : z \in \partial S\} = \sqrt{13}$$

and hence

$$\max\left\{|z^3 - 1| : 0 \le \operatorname{Re} z \le 1,\ 0 \le \operatorname{Im} z \le 1\right\} = \sqrt{13}.$$

4.2 (a) Since f is non-zero on $\overline{\mathcal{R}}$, the function $g = 1/f$ is analytic on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$. Hence, by the Maximum Principle, there is a point α on $\partial\mathcal{R}$ such that

$$|g(z)| \le |g(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}},$$

and hence (because $|g| = |1/f| = 1/|f|$)

$$|f(z)| \ge |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}},$$

as required.

(b) Since $f(z) = \exp\left(z^2\right)$ is analytic on $\{z : |z| < 1\}$, and continuous and non-zero on $\{z : |z| \le 1\}$, it follows from the Minimum Principle that there exists $\alpha \in \partial D = \{z : |z| = 1\}$ such that

$$\min\left\{\left|\exp\left(z^2\right)\right| : |z| \le 1\right\} = |f(\alpha)|.$$

Since each point of ∂D has the form e^{it}, for $t \in [0, 2\pi]$, we need to determine

$$\min\left\{\left|\exp\left(e^{i2t}\right)\right| : 0 \le t \le 2\pi\right\}.$$

Now $\left|\exp\left(e^{i2t}\right)\right| = \exp(\cos 2t)$, and so

$$\begin{aligned} \min\left\{\left|\exp\left(e^{i2t}\right)\right| : 0 \le t \le 2\pi\right\} &= \min\{\exp(\cos 2t) : 0 \le t \le 2\pi\} \\ &= e^{-1}, \end{aligned}$$

obtained when $\cos 2t = -1$, which corresponds to $\alpha = e^{i\pi/2} = i$ and $\alpha = e^{i3\pi/2} = -i$. Thus

$$\min\left\{\left|\exp\left(z^2\right)\right| : |z| \le 1\right\} = e^{-1}.$$

4.3 Since $h = f - g$ is analytic on $\mathcal{R}$ and continuous on $\overline{\mathcal{R}}$, $|h| = |f - g|$ attains its maximum on $\overline{\mathcal{R}}$ at some point of $\partial\mathcal{R}$. Since $f = g$ on $\partial\mathcal{R}$, this maximum value is 0. Hence $|h| = |f - g| = 0$ on $\mathcal{R}$; that is, $f = g$ on $\mathcal{R}$.

4.4 Since $f(D) = D$, where $D = \{z : |z| < 1\}$,

$$|f(z)| < 1, \quad \text{for } |z| < 1.$$

Since f is analytic on D with $f(0) = 0$, it follows from Schwarz's Lemma with $M = 1$ and $R = 1$ that

$$|f(z)| \le |z|, \quad \text{for } |z| < 1. \tag{1}$$

Since f is one-one and analytic on D, f^{-1} exists and is analytic on $f(D) = D$ (by the Inverse Function Rule). Also $f^{-1}(0) = 0$ (since $f(0) = 0$). Hence, replacing f by f^{-1} in the argument leading to Inequality (1), we obtain

$$|f^{-1}(z)| \le |z|, \quad \text{for } |z| < 1. \tag{2}$$

Replacing z by $f(z)$ in Inequality (2) gives

$$|f^{-1}(f(z))| \le |f(z)|, \quad \text{for } |f(z)| < 1,$$

that is,

$$|z| \le |f(z)|, \quad \text{for } |z| < 1. \tag{3}$$

From Inequalities (1) and (3), $|f(z)| = |z|$, for $|z| < 1$, so that the function

$$g(z) = f(z)/z \quad (z \in D - \{0\})$$

is analytic on $D - \{0\}$ and satisfies

$$|g(z)| = 1, \quad \text{for } z \in D - \{0\}. \tag{4}$$

Now we show that g is a constant function. It follows from Equation (4) that g has a local maximum at each point of $D - \{0\}$. Hence g is constant on $D - \{0\}$, that is,

$$g(z) = \lambda, \quad \text{for } z \in D - \{0\},$$

where $|\lambda| = 1$. Hence

$$f(z) = \lambda z, \quad \text{for } z \in D - \{0\},$$

and this equality also holds for $z = 0$, giving the desired result.

SOLUTIONS TO THE EXERCISES

Section 1

1.1 (a) $\mathrm{Wnd}(\Gamma, \alpha_1) = 2$, $\mathrm{Wnd}(\Gamma, \alpha_2) = 1$, $\mathrm{Wnd}(\Gamma, \alpha_3) = 0$.

(b) Since

$$2 + \cos \tfrac{3}{2}t \geq 2 - 1 = 1 > 0, \quad \text{for } 0 \leq t \leq 4\pi,$$

we have

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{it}, \quad \text{for } 0 \leq t \leq 4\pi.$$

Hence a continuous argument function is

$$\theta(t) = t \quad (t \in [0, 4\pi]).$$

Thus

$$\mathrm{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(4\pi) - \theta(0)) = 2.$$

1.2 Since

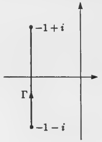
$$\gamma(t) \in \mathbb{C}_{2\pi}, \quad \text{for } -1 \leq t \leq 1,$$

and $\mathrm{Arg}_{2\pi}$ is continuous on $\mathbb{C}_{2\pi}$, we deduce that

$$\theta(t) = \mathrm{Arg}_{2\pi}(\gamma(t)) \quad (t \in [-1, 1])$$

is a continuous argument function for Γ. Thus

$$\begin{aligned}\mathrm{Wnd}(\Gamma, 0) &= \frac{1}{2\pi}(\theta(1) - \theta(-1)) \\ &= \frac{1}{2\pi}(\mathrm{Arg}_{2\pi}(-1+i) - \mathrm{Arg}_{2\pi}(-1-i)) \\ &= \frac{1}{2\pi}\left(\frac{3\pi}{4} - \frac{5\pi}{4}\right) = -\frac{1}{4}.\end{aligned}$$



1.3 (a)

$$\begin{aligned}\mathrm{Log}_{3\pi}(-i) &= \log_e |-i| + i\,\mathrm{Arg}_{3\pi}(-i) \\ &= 0 + i\frac{3\pi}{2},\end{aligned}$$

since $3\pi/2$ is an argument of $-i$ and $\pi < 3\pi/2 \leq 3\pi$.

(b)

$$\begin{aligned}\mathrm{Log}_{2\pi}(2) &= \log_e 2 + i\,\mathrm{Arg}_{2\pi}(2) \\ &= \log_e 2 + 2\pi i,\end{aligned}$$

since 2π is an argument of 2 and $0 < 2\pi \leq 2\pi$.

(c)

$$\begin{aligned}\mathrm{Log}_{3\pi/2}(-3) &= \log_e |-3| + i\,\mathrm{Arg}_{3\pi/2}(-3) \\ &= \log_e 3 + \pi i,\end{aligned}$$

since π is an argument of -3 and $-\pi/2 < \pi \leq 3\pi/2$.

1.4 By Problem 1.8,

$$\int_\Gamma \frac{1}{z-2}\,dz = 2\pi i\,\mathrm{Wnd}(\Gamma, 2),$$

since Γ in Exercise 1.1 is closed and $2 \notin \Gamma$. From the diagram in Exercise 1.1, $\mathrm{Wnd}(\Gamma, 2) = 1$, so that

$$\int_\Gamma \frac{1}{z-2}\,dz = 2\pi i \times 1 = 2\pi i.$$

Section 2

2.1 (a) Since f has a zero of order three at the point 1, we deduce by Theorem 2.1 that f'/f has a simple pole at 1 with $\mathrm{Res}(f'/f, 1) = 3$. Similarly, f'/f has simple poles at 2 and 3 with

$$\mathrm{Res}(f'/f, 2) = 2 \quad \text{and} \quad \mathrm{Res}(f'/f, 3) = 1.$$

(b) By the Argument Principle, $\mathrm{Wnd}(f(\Gamma), 0)$ is the number of zeros of f inside Γ, counted according to their orders. Since f has $3 + 2 + 1 = 6$ zeros inside Γ, counted according to their orders, we have $\mathrm{Wnd}(f(\Gamma), 0) = 6$.

2.2 (a) Since $\mathrm{Wnd}\big(f(\Gamma), -\frac{1}{2}\big) = 1$, the equation $f(z) = -\frac{1}{2}$ has one solution inside Γ.

(b) Since $\mathrm{Wnd}\big(f(\Gamma), \frac{1}{4}\big) = 2$, the equation $f(z) = \frac{1}{4}$ has two solutions inside Γ.

(c) Since $\mathrm{Wnd}(f(\Gamma), 2i) = 0$, the equation $f(z) = 2i$ has no solutions inside Γ.

2.3 (a) (i) We let $\Gamma_1 = \partial S_1 = \{z : |z| = 2\}$ and choose a dominant term on Γ_1. If $g(z) = z^5$, then $f(z) - g(z) = 3z + 10$ and, for $z \in \Gamma_1$,

$$\begin{aligned}|f(z) - g(z)| &= |3z + 10| \\ &\leq 3|z| + 10 \quad \text{(Triangle Inequality)} \\ &\leq 16,\end{aligned}$$

whereas

$$|g(z)| = |z|^5 = 32 > 16, \quad \text{for } z \in \Gamma_1.$$

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ_1, namely five, arising from the zero of order five of g at 0.

(ii) We let $\Gamma_2 = \partial S_2 = \{z : |z| = 1\}$ and choose a dominant term on Γ_2. If $g(z) = 3z + 10$, then $f(z) - g(z) = z^5$ and, for $z \in \Gamma_2$,

$$|f(z) - g(z)| = |z^5| = |z|^5 = 1,$$

whereas

$$\begin{aligned}|g(z)| &= |3z + 10| \\ &\geq \big||3z| - 10\big| \quad \text{(Triangle Inequality)} \\ &= 7 > 1, \quad \text{for } z \in \Gamma_2.\end{aligned}$$

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ_2, namely none.

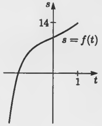
Remark We could have chosen $g(z) = 10$ here. Then, as you can check, for $z \in \Gamma_2$,

$$|f(z) - g(z)| \leq 4 \quad \text{and} \quad |g(z)| = 10 > 4.$$

Hence, by Rouché's Theorem, f has no zeros inside Γ_2 since $g(z) = 10$ has none.

(iii) Since f has no zeros on Γ_2 (because, from part(ii), $|f(z) - g(z)| < |g(z)|$, for $z \in \Gamma_2$) and no zeros in S_2, we deduce that f has five zeros in S_3.

(iv) Since f has exactly one real zero (because it is strictly increasing: see the figure) and $f(\overline{z}) = \overline{f(z)}$, we deduce that the other four zeros of f form two complex conjugate pairs. Hence f has two zeros in S_4.



(b) We let $\Gamma = \partial S = \{z : |z| = \frac{1}{2}\}$ and choose a dominant term on Γ. If $g(z) = 3z$, then $f(z) - g(z) = \mathrm{Log}(1+z)$ and, for $z \in \Gamma$,

$$\begin{aligned}|f(z) - g(z)| &= |\mathrm{Log}(1+z)| \\ &\le 2|z| \quad \text{(by the given estimate)} \\ &= 1,\end{aligned}$$

whereas

$$|g(z)| = |3z| = \tfrac{3}{2} > 1, \quad \text{for } |z| = \tfrac{1}{2}.$$

Hence, by Rouché's Theorem, f has the same number of zeros as g inside Γ, namely one, arising from the simple zero of g at 0.

Section 3

3.1 If f is non-constant and analytic on $D = \{z : |z| < 1\}$, then $f(D)$ is open by the Open Mapping Theorem. However, the condition

$$|f(z)| = \pi, \quad \text{for } |z| < 1,$$

implies that $f(D) \subseteq \{w : |w| = \pi\}$, and hence that $f(D) = \varnothing$. Since $f(D) \neq \varnothing$, we deduce that f must be constant.

3.2 Since $f(\alpha) = f\left(\frac{1}{2}\right) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$, we have

$$\begin{aligned}f(z) - f(\alpha) = z - z^2 - \tfrac{1}{4} &= -\left(z^2 - z + \tfrac{1}{4}\right) \\ &= -\left(z - \tfrac{1}{2}\right)^2 \\ &= \left(i\left(z - \tfrac{1}{2}\right)\right)^2.\end{aligned}$$

Hence

$$f(z) = f(\alpha) + (\phi(z))^2, \quad \text{for } z \in \mathbb{C},$$

where $\phi(z) = i\left(z - \frac{1}{2}\right)$.

Since ϕ is a translation by $-\frac{1}{2}$, followed by an anticlockwise rotation through $\pi/2$, we deduce that ϕ is analytic and one-one on $\mathbb{C}$. Hence f satisfies the definition of two-one near $\alpha = \frac{1}{2}$.

3.3 (a) The Taylor series about $\alpha = 0$ for f is

$$f(z) = \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots.$$

Hence f is two-one near 0, by the Local Mapping Theorem.

(b) Since $f'(2\pi i) = e^{2\pi i} = 1 \neq 0$, we deduce that f is one-one near $2\pi i$.

(c) The Taylor series about 0 for f is

$$f(z) = \sin z - z = -\frac{z^3}{3!} + \frac{z^5}{5!} - \cdots.$$

Hence f is three-one near 0, by the Local Mapping Theorem.

3.4 (a) The Taylor series about $\alpha = 0$ for f is

$$f(z) = 3z + z^3.$$

Since f is odd, the Taylor series for f^{-1} about $\beta = f(\alpha) = 0$ is of the form

$$f^{-1}(w) = b_1 w + b_3 w^3 + b_5 w^5 + \cdots,$$

where $b_1, b_3, \ldots$ satisfy

$$z = b_1(3z + z^3) + b_3(3z + z^3)^3 + b_5(3z + z^3)^5 + \cdots.$$

Equating coefficients of powers of z, we obtain

$$\begin{aligned}&z: && 1 = 3b_1 && \Longrightarrow b_1 = \tfrac{1}{3} \\ &z^3: && 0 = b_1 + 27b_3 && \Longrightarrow b_3 = -b_1/27 = -\tfrac{1}{81} \\ &z^5: && 0 = 27b_3 + 243b_5 && \Longrightarrow b_5 = -b_3/9 = \tfrac{1}{729}.\end{aligned}$$

Hence

$$f^{-1}(w) = \tfrac{1}{3}w - \tfrac{1}{81}w^3 + \tfrac{1}{729}w^5 - \cdots.$$

(b) The Taylor series about $\alpha = 1$ for f is

$$\begin{aligned}f(z) = e^z &= e e^{z-1} \\ &= e\left(1 + (z-1) + \frac{(z-1)^2}{2!} + \cdots\right).\end{aligned}$$

Thus the Taylor series for f^{-1} about $\beta = f(\alpha) = e$ is of the form

$$f^{-1}(w) = b_0 + b_1(w - e) + b_2(w - e)^2 + \cdots,$$

where $b_0 = \alpha = 1$ and $b_1, b_2, \ldots$ satisfy

$$\begin{aligned}z - 1 = b_1&\left(e(z-1) + \frac{e(z-1)^2}{2!} + \cdots\right) \\ &+ b_2\left(e(z-1) + \frac{e(z-1)^2}{2!} + \cdots\right)^2 + \cdots.\end{aligned}$$

Equating coefficients of powers of $z - 1$, we obtain

$$\begin{aligned}&z - 1: && 1 = b_1 e && \Longrightarrow b_1 = 1/e \\ &(z-1)^2: && 0 = \tfrac{1}{2}b_1 e + b_2 e^2 && \Longrightarrow b_2 = -\tfrac{1}{2}b_1/e = -\tfrac{1}{2}/e^2.\end{aligned}$$

Hence

$$f^{-1}(w) = 1 + \frac{1}{e}(w - e) - \frac{1}{2e^2}(w - e)^2 + \cdots.$$

Section 4

4.1 (a) In this part there is no need to use the Maximum Principle. For $|z| \le 1$, we have

$$\begin{aligned}|z^2 + 2| &\le |z|^2 + 2 \quad \text{(Triangle Inequality)} \\ &\le 3.\end{aligned}$$

Also, if $z = 1$, then

$$|z^2 + 2| = 1 + 2 = 3.$$

We deduce that

$$\max\left\{|z^2 + 2| : |z| \le 1\right\} = 3.$$

This maximum is attained when $z^2 = 1$, that is, for $z = \pm 1$.

(b) Since the function $f(z) = z^2 - 2$ is analytic on the open disc $D = \{z : |z - i| < 1\}$ and continuous on $\overline{D} = \{z : |z - i| \le 1\}$, it follows from the Maximum Principle that there exists $\alpha \in \partial D = \{z : |z - i| = 1\}$ such that

$$\max\{|f(z)| : |z - i| \le 1\} = |f(\alpha)|.$$

Since each point of ∂D has the form $i + e^{it}$, for some $t \in [0, 2\pi]$, we need to determine

$$\max\left\{\left|f\left(i + e^{it}\right)\right| : 0 \le t \le 2\pi\right\}.$$

Because

$$\begin{aligned} f(i+e^{it}) &= (i+e^{it})^2 - 2 \\ &= -3 + 2ie^{it} + e^{2it} \\ &= (-3 - 2\sin t + \cos 2t) + i(2\cos t + \sin 2t), \end{aligned}$$

we obtain

$$\begin{aligned} |f(i+e^{it})|^2 &= (-3 - 2\sin t + \cos 2t)^2 + (2\cos t + \sin 2t)^2 \\ &= \big(9 + 4\sin^2 t + \cos^2 2t \\ &\qquad +12\sin t - 6\cos 2t - 4\sin t\cos 2t\big) \\ &\quad + \big(4\cos^2 t + 4\cos t \sin 2t + \sin^2 2t\big) \\ &= 14 + 4(\sin 2t\cos t - \sin t\cos 2t) \\ &\quad + 12\sin t - 6\cos 2t \\ &= 14 + 16\sin t - 6(1 - 2\sin^2 t) \\ &= 8 + 16\sin t + 12\sin^2 t \\ &= 12\left(\tfrac{2}{3} + \tfrac{4}{3}\sin t + \sin^2 t\right) \\ &= 12\left(\tfrac{2}{9} + \left(\tfrac{2}{3} + \sin t\right)^2\right). \end{aligned}$$

The maximum of this expression is 36, obtained when $\frac{2}{3} + \sin t = \frac{5}{3}$, that is, when $\sin t = 1$, which corresponds to $\alpha = i + e^{it} = 2i$.

Thus

$$\max\left\{\left|f\left(i+e^{it}\right)\right| : 0 \le t \le 2\pi\right\} = \sqrt{36} = 6,$$

and so

$$\max\left\{|z^2 - 2| : |z - i| \le 1\right\} = 6.$$

(c) In this part there is no need to use the Maximum Principle. For $z = x + iy$, we have

$$\begin{aligned} |e^{z^2}| &= e^{\operatorname{Re}(z^2)} \\ &= e^{x^2 - y^2}. \end{aligned}$$

Now, if $-1 \le x \le 1$ and $-1 \le y \le 1$, then $x^2 - y^2 \le 1$, so that

$$|e^{z^2}| \le e^1 = e.$$

Also, if $z = 1$, then

$$|e^{z^2}| = e^1 = e.$$

We deduce that

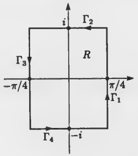
$$\max\left\{|e^{z^2}| : -1 \le \operatorname{Re} z \le 1, -1 \le \operatorname{Im} z \le 1\right\} = e.$$

This maximum is attained when $x^2 - y^2 = 1$, that is, for $z = \pm 1$.

(d) Since $f(z) = \tan z$ is analytic on the open rectangle $R = \{z : -\pi/4 < \operatorname{Re} z < \pi/4, -1 < \operatorname{Im} z < 1\}$ and continuous on $\overline{R}$, it follows from the Maximum Principle that there exists $\alpha \in \partial R$ such that

$$\max\{|f(z)| : z \in \overline{R}\} = |f(\alpha)|.$$

Thus we need only find the maximum value of $|f(z)|$, for $z \in \partial R$. Let $\partial R = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$ as in the figure.



Also, since $f(z) = \tan z$ is odd we need only consider the values of $|f(z)|$ on Γ_1 and Γ_2.

Now, if $z = x + iy$, then

$$\begin{aligned} |\tan z|^2 &= \frac{|\sin z|^2}{|\cos z|^2} \\ &= \frac{\sin^2 x + \sinh^2 y}{\cos^2 x + \sinh^2 y}, \end{aligned}$$

by *Unit A2*, Example 4.4 and Problem 4.7.

Thus, for $z \in \Gamma_1$, we have

$$|\tan z|^2 = \frac{\frac{1}{2} + \sinh^2 y}{\frac{1}{2} + \sinh^2 y} = 1, \quad \text{for } -1 \le y \le 1,$$

since $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$. On the other hand, for $z \in \Gamma_2$, we have

$$\begin{aligned} |\tan z|^2 &= \frac{\sin^2 x + \sinh^2 1}{\cos^2 x + \sinh^2 1} \\ &\le \frac{\frac{1}{2} + \sinh^2 1}{\frac{1}{2} + \sinh^2 1} = 1, \end{aligned}$$

since sin is increasing on $[0, \pi/4]$ and cos is decreasing on $[0, \pi/4]$. Hence

$$\max\{|\tan z| : z \in \partial R\} = 1,$$

so that

$$\max\{|\tan z| : z \in \overline{R}\} = 1,$$

with the maximum attained at all points on Γ_1 and Γ_3.

4.2 **(a)** In the proof of Schwarz's Lemma we found that

$$\left|\frac{f(z)}{z}\right| \le \frac{M}{R}, \quad \text{for } 0 < |z| < R.$$

Since

$$f'(0) = \lim_{z \to 0} \frac{f(z)}{z},$$

we deduce that

$$|f'(0)| = \lim_{z \to 0} \left|\frac{f(z)}{z}\right| \le \frac{M}{R},$$

as required.

(b) Define the analytic function g as in the proof of Schwarz's Lemma. If

$$\left|\frac{f(z_0)}{z_0}\right| = \frac{M}{r}, \quad \text{for some } z_0 \text{ with } 0 < |z_0| < R,$$

then the function g must have a local maximum at z_0, which is possible only if g is constant, say $g(z) = \lambda$ for $0 < |z| < R$. Hence $f(z) = \lambda z$, for $|z| < R$, where $|\lambda| = M/R$, as required.

If $|f'(0)| = M/R$, then $|g(0)| = |f'(0)| = M/R$, so that the function g has a local maximum at 0, which is again possible only if g is constant. Thus, once again, $f(z) = \lambda z$, for $|z| < R$, where $|\lambda| = M/R$.